Instructions: *You must work two problems from Section A, two from Section B, and one additional problem from either of the two sections (for a total of *five* problems).* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

Section A:

A1 Let $f : X \to Y$ be a continuous function between topological spaces X and Y, and let A be a subset of X

(a) If A is compact, show that $f(A)$ is compact.

(b) If A is connected, show that $f(A)$ is connected.

A2 Let $X = Y \cup Z$, where

$$\begin{aligned} Y =& \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}, \text{ and} \\ Z =& \{(x, \sin(1/x)) \mid 0 < x \leq 1\} \end{aligned}$$

Prove that X is connected but not path connected.

A3 Let X and Y be topological spaces, and let $x_0 \in X$ and $y_0 \in Y$. Show that

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

A4 Let A be a subspace of topological space X. A continuous map $r : X \to A$ is said to be a *retraction* if $r(a) = a$ for ever $a \in A$.

(a) Show that if $r : X \to A$ is a retraction, then the induced map $r_* : \pi_1(X, x_0) \to \pi_1(A, a_0)$ is surjective. Conclude that there does not exist a retraction $r : D^2 \to S^1$.

(b) Prove that every continuous map : $D^2 \to D^2$ has a fixed point.

Section B:

B1 Let $p : E \to B$ be a covering map with connected bases space B, and let x and y be two points in B. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

B2 Let $X = S^1 \vee S^1$ be the wedge of two circles. Compute the fundamental group of X from the fact that the fundamental group of the circle S^1 is isomorphic to $\mathbb{Z}$. Give examples of normal and non-normal 3–fold covers of X (justify your examples).

B3 Let X be a path-connected and locally path-connected space with finite fundamental group. Show that any map $f : X \to S^1$ is null-homotopic.

B4 Show that the following three conditions are equivalent

(a) Every continuous map $f : S^1 \to X$ is null-homotopic.

(b) Every continuous map $f : S^1 \to X$ extends to a map $\tilde{f} : D^2 \to X$.

(c) The fundamental group $\pi_1(X, x_0)$ is trivial for every point $x_0 \in X$.
