Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let X and Y be topological spaces, and $f : X \longrightarrow Y$ a function. Recall that *"f is open"* means: If U is an open set in X then $f(U)$ is open in Y.

i. If f is continuous, does it follow that f is open? (Proof or counterexample.)

ii. If f is open, does it follow that f is continuous? (Proof or counterexample.)

iii. Show that if $X = Y \times Z$ (i.e. a product of topological spaces, with the product topology), and if f is the projection on Y, then f is open.

A2. Prove that the product of finitely many connected spaces is connected.

A3. Let X be a topological space with the property that for any two distinct points of X, there is an open set containing exactly one of them. Suppose also that for any $x \in X$ and closed subset A of X not containing x, there are disjoint open sets $U \ni x$ and $V \subseteq A$. Prove that X is Hausdorff.

B. **Homotopy** (2 problems)

B1. Let $p : E \to B$ be a covering map with connected base space B, and let x and y be two points in B. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

B2. Construct a space that has the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as fundamental group (explain).

B3. Let X be a path-connected space. Prove that $\pi_1(X, x)$ is isomorphic to $\pi_1(X, y)$ for all points $x, y \in X$.

C. **Mixed** (1 problem)

C1. Show that the following three conditions on a topological space X are equivalent:

i. Every continuous map $S^1 \longrightarrow X$ is null homotopic.

ii. Every continuous map $S^1 \longrightarrow X$ extends to a continuous map $D^2 \longrightarrow X$.

iii. The fundamental group $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

C2. Prove the Brouwer fixed point theorem in dimensions one and two: Every continuous map $f : D^n \longrightarrow D^n$, $n = 1, 2$, has a fixed point.
