Instructions: *Work two problems from section A, two problems from section B, and one additional problem from either section, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

Section A.

A1 Let $p\colon X \to Y$ be a quotient map of topological spaces, such that Y is connected and each set $p^{-1}(y)$, $y \in Y$, is a connected subspace of X. Prove that X is connected.

A2 Let $\{ X_\alpha \mid \alpha \in J \}$ be a family of topological spaces, and let $X = \prod_{\alpha\in J} X_\alpha$ with the product topology. Let $\pi_\alpha\colon X \to X_\alpha$ be the projection, and let Y be a topological space. Prove that a function $f\colon Y \to X$ is continuous if and only if the composite $\pi_\alpha \circ f$ is continuous for each $\alpha \in J$.

A3 (a) Let X be a Hausdorff space and A a compact subset of X. Prove that A is closed.

(b) Let $f\colon X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y. Prove that f is a closed map.

A4 Prove that a product (finite or infinite) of path-connected spaces is path-connected.

Section B.

B1 Let X be the subspace of $\mathbb{R}^2$ that is the union of two circles of radius 1 centered at $(-2, 0)$ and $(2, 0)$ and the line segment from $(-1, 0)$ to $(1, 0)$. State the Seifert-van Kampen Theorem, and use it to find the fundamental group of X.

B2 Let $p\colon E \to B$ be a covering map. Suppose that B is path-connected and x and y are points of B. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

B3 Show that a retract of a contractible space is contractible.

B4 Give an example of a space whose fundamental group is a cyclic group of order five. Provide a proof that the fundamental group is cyclic of order five.