**Instructions**: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

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A. **Point Set Topology** (2 problems)

A1. Let $X$ and $Y$ be topological spaces.

i. Prove that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.

ii. Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Prove that $f$ is continuous if and only if $f_1$ and $f_2$ are continuous.

A2. Let $f : X \to Y$ be a quotient map of topological spaces, such that $Y$ is connected and for each $y \in Y$, the set $f^{-1}(y)$ is a connected subspace of $X$. Show that $X$ is connected.

A3. i. Prove that a compact subset of a Hausdorff space is closed.

ii. Let $f : X \to Y$ be a continuous map from a compact space $X$ to a Hausdorff space $Y$. Prove that $f$ is a closed map.

B. **Homotopy** (2 problems)

B1. Show that if a path-connected, locally path-connected space $X$ has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

B2. Compute the fundamental group of $S^2$ with $n$ points removed, where $n$ is a positive integer. Also compute the fundamental group of $\mathbb{R}^3$ with the three coordinate axes removed.

B3. Let $T^2$ be the torus $S^1 \times S^1$ and let $M_{1,1}$ denote the surface with genus 1 and one boundary component (shown below). Let $X$ be the space obtained from the disjoint union of $T^2$ and $M_{1,1}$ by identifying a curve $S^1 \times \{x_0\}$ in $T^2$ with the boundary curve of $M_{1,1}$. State the Seifert–van Kampen Theorem and use it to compute the fundamental group of $X$.

C. **Mixed** (1 problem)

C1. Prove that no two of the spaces $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are homeomorphic.

C2. A space $X$ is said to be *contractible* if the identity map of $X$ to itself is null homotopic.

i. Show that $\mathbb{R}$ is contractible.

ii. Show that a contractible space is path connected.

iii. Show that if $X$ is contractible, and $Y$ is path connected, then $[X, Y]$ has a single element, where $[X, Y]$ denotes the set of homotopy classes of maps of $X$ into $Y$.

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