Mathematics Comprehensive Examination
Core II - Algebra (Galois Theory)
August 2002

Directions: Do the first two problems, marked (*), and two additional problems for a total of four. Only turn in these four problems! Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet. You will be graded on the clarity as well as on the correctness of your response. You have 3 and 1/2 hours for this test. Good luck!

1*. If $f(x) \in K[x]$ is irreducible of degree $d \geq 1$
and if $L|K$ is a finite extension of degree n with $gcd(n, d) = 1$,
then f is irreducible in $L[x]$.

2*. Prove or disprove: If $[K : Q] = 3$ and if $[L : Q] = 3$ and if
$L \cap K = Q$ then the degree of $L \cdot K$ over K is 9. Here, $L \cdot K$ denots the compositum of L and K, that is, the smallest subfield of $\overline{Q}$
containing both L and K.

3. Let $f(x)$ be a separable polynomial of degree n that is irreducible in $K[x]$, where K is a field. Let α be a root of $f(x)$ in a field L containing K.
Let r denote the number of roots of $f(x)$ in $K[\alpha]$.
Prove that r divides n.
Hint: Where are the remaining $n - r$ roots of $f(x)$ not in $K[\alpha]$?

4. Give an explicit generator for each subfield of the splitting field of $x^4 + x^3 + x^2 + x + 1$ over Q.

5. Prove or disprove: If $L|K$ is a finite galois extension and if $K|F$ is a finite galois extension, then $L|F$ is a finite galois extension.

6. Let C_n denote a cyclic group of order n. Prove that there is a galois extension $K|Q$ whose Galois group is isomorphic to C_n.
You may use Dirichlet's theorem stating that the arithmetic
progression $1, 1 + n, 1 + 2n, 1 + 3n, 1 + 4n, \cdots$
contains infinitely mahy prime numbers.