Mathematics Comprehensive Exam
Core II - Algebra
August 2003

Directions: Do the first two problems, marked (*). Then do an additional two problems for a total of four. Start each problem on a new sheet of paper. Put your name and problem number at the top of every sheet. You will be graded for clarity as well as correctness of your responses. You have 3 and 1/2 hours for the test. Good luck!

1*. a) Let G be any group, and let $X = G$. Define a function

$$G \times X \to X : \ (g, x) \to gxg^{-1}$$

(we abbreviate this by $g.x := gxg^{-1}$). Check that this defines an action of the group G on the set X. This is the conjugation action of G on itself.

b) Let $G = S_4$ be the symmetric group on four letters. Determine the orbits of G acting on itself by conjugation. For each orbit, choose a representative x, and determine the stabilizer

$$G_x = \{g \in G \mid g.x = x\}$$

2*. Let F be a field. Let $\overline{F}$ be an algebraic closure of F, and $\alpha, \beta \in \overline{F}$. Let $F(\alpha), F(\beta)$ be the subfields of $\overline{F}$ generated by these elements and F. Show that there is an isomorphism of fields $\varphi : F(\alpha) \to F(\beta)$, such that $\varphi \mid F =$identity, if and only if α and β satisfy the same irreducible polynomial $f(x) \in F[x]$.

3. Let $\mathbf{F}_3$ be the field with 3 elements. Show that $x^3 - x + 1$ is irreducible in $\mathbf{F}_3[x]$. Let α be any root of this polynomial and consider the field $\mathbf{F}_3(\alpha)$. This has 27 elements.

a) Find expressions in the form $a + b\alpha + c\alpha^2$, with $a, b, c \in \mathbf{F}_3$ for α^4 and $1/(\alpha + 1)$.

b) Find expressions as in part a) for the other two roots β, γ of $x^3 - x + 1 = 0$. Hint: Use the Galois group of $\mathbf{F}_{27}/\mathbf{F}_3$.

4. a) Show that the polynomial $x^3 - x - 1$ is irreducible in $\mathbf{Q}[x]$.

b) Determine $Gal(E/\mathbf{Q})$ where E is the splitting field of of $x^3 - x - 1$. Justify your answer. The discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$. Determine all the intermediate fields $\mathbf{Q} \subset K \subset E$.

5) a) Let F be a field and let $f(x) \in F[x]$ be a polynomial. Show that $f(x)$ has multiple roots (in some extension field of F) if and only if $f(x)$ and its derivative $f'(x)$ have a common root.

b) Give an example of an *irreducible* $f(x)$ which has multiple roots.

6) a) Consider the 7th cyclotomic polynomial

$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

On substitution $x \mapsto x + 1$ this becomes

$$x^6 + 7x^5 + 35x^4 + 35x^3 + 21x^2 + 21x + 7.$$

Use this information to show that $\Phi_7(x)$ is irreducible in $\mathbf{Q}[x]$. Hint: Eisenstein's criterion.

b) What is the Galois group of $E/\mathbf{Q}$ where E is the splitting field of $\Phi_7(x)$. Draw the lattice of subfields of E.