Mathematics Comprehensive Examination
Algebra Core II (Field Theory and Galois Theory)
August 2005

Do problem 1 and any three additional problems, for a total of four. The test is intended to be 3 hours long; you will also have 30 minutes overtime, for a total of 3 and a half hours.

1.* If $f \in K[x]$ is irreducible of degree $d \geq 1$, and L/K is a finite extension of degree n with $\gcd(n, d) = 1$, then f is irreducible in $L[x]$.

2. (a) Let K and L be fields, and let $\phi : K \to L$ be a map that is additive and multiplicative:

$$\phi(a + b) = \phi(a) + \phi(b) \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b),$$

for all $a, b \in K$. Prove that if ϕ is not the identically-0 map, then $\phi(1_K) = 1_L$.

(b) Let L be a field containing the rational numbers. Prove that the only injective ring homomorphism from $\mathbb{Q}$ into L is the identity map (= the inclusion map $\mathbb{Q} \hookrightarrow L$). (You may use the fact that the rational number 1 is the multiplicative identity of L; i.e., $1_{\mathbb{Q}} = 1_L$.)

(c) Prove that the only field automorphism of $\mathbb{R}$ is the identity. You may use the fact that a real number is nonnegative if and only if it is the square of a real number. (Hint: How would you represent a real number in terms of rational numbers?)

3. (a) Let F be a field and let $f \in F[x]$ be a polynomial. Show that f has multiple roots (in some extension field of F) if and only if f and its derivative f' have a common root.

(b) Give an example of an *irreducible* f having a multiple root.

4. Let K denote the splitting field, over $\mathbb{Q}$, of the polynomial $X^3 - 2$.

(a) What is the discriminant of $x^3 - 2$?

(b) Determine the Galois group of K over $\mathbb{Q}$, and all intermediate fields of this extension, and set up the Galois correspondence between intermediate fields and subgroups of the Galois group.

5. Let f be an irreducible polynomial of degree 6 over a field F. Let K be an extension field of F with $[K : F] = 2$. Prove that if f is reducible over K, then it factors in $K[x]$ into the product of two irreducible cubic polynomials.

6. Let $k \subseteq E \subseteq K$ be fields, with E a finite extension of k and K a finite extension of E. Prove that K is a finite extension of k, and $[K : k] = [K : E][E : k]$.

7. Let p be a prime, and $n \in \mathbb{N}$; write $q = p^n$. Show that for every (positive) divisor d of n, the finite field $\mathbb{F}_q$ has exactly one subfield of order p^d. Show also that for every $d \in \mathbb{N}$ not dividing n, $\mathbb{F}_q$ has no subfield of order p^d.