Do problem 1 and any three additional problems, for a total of four. The test is intended to be 3 hours long; you will also have 30 minutes overtime, for a total of 3 and a half hours.

1. * If \( f \in K[x] \) is irreducible of degree \( d \geq 1 \), and \( L/K \) is a finite extension of degree \( n \) with \( \gcd(n, d) = 1 \), then \( f \) is irreducible in \( L[x] \).

2. (a) Let \( K \) and \( L \) be fields, and let \( \phi : K \to L \) be a map that is additive and multiplicative:
\[
\phi(a + b) = \phi(a) + \phi(b) \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b),
\]
for all \( a, b \in K \). Prove that if \( \phi \) is not the identically-0 map, then \( \phi(1_K) = 1_L \).

(b) Let \( L \) be a field containing the rational numbers. Prove that the only injective ring homomorphism from \( \mathbb{Q} \) into \( L \) is the identity map (= the inclusion map \( \mathbb{Q} \hookrightarrow L \)). (You may use the fact that the rational number 1 is the multiplicative identity of \( L \); i.e., \( 1_{\mathbb{Q}} = 1_L \).)

(c) Prove that the only field automorphism of \( \mathbb{R} \) is the identity. You may use the fact that a real number is nonnegative if and only if it is the square of a real number. (Hint: How would you represent a real number in terms of rational numbers?)

3. (a) Let \( F \) be a field and let \( f \in F[x] \) be a polynomial. Show that \( f \) has multiple roots (in some extension field of \( F \)) if and only if \( f \) and its derivative \( f' \) have a common root.

(b) Give an example of an irreducible \( f \) having a multiple root.

4. Let \( K \) denote the splitting field, over \( \mathbb{Q} \), of the polynomial \( x^3 - 2 \).

(a) What is the discriminant of \( x^3 - 2 \)?

(b) Determine the Galois group of \( K \) over \( \mathbb{Q} \), and all intermediate fields of this extension, and set up the Galois correspondence between intermediate fields and subgroups of the Galois group.

5. Let \( f \) be an irreducible polynomial of degree 6 over a field \( F \). Let \( K \) be an extension field of \( F \) with \( [K : F] = 2 \). Prove that if \( f \) is reducible over \( K \), then it factors in \( K[x] \) into the product of two irreducible cubic polynomials.

6. Let \( k \subseteq E \subseteq K \) be fields, with \( E \) a finite extension of \( k \) and \( K \) a finite extension of \( E \). Prove that \( K \) is a finite extension of \( k \), and \( [K : k] = [K : E][E : k] \).

7. Let \( p \) be a prime, and \( n \in \mathbb{N} \); write \( q = p^n \). Show that for every (positive) divisor \( d \) of \( n \), the finite field \( \mathbb{F}_q \) has exactly one subfield of order \( p^d \). Show also that for every \( d \in \mathbb{N} \) not dividing \( n \), \( \mathbb{F}_q \) has no subfield of order \( p^d \).