Mathematics Comprehensive Examination
Algebra Core II (Field Theory and Galois Theory)
August 2006

Do problem 1 and any three additional problems, for a total of four. The test is intended to be 3 hours long; you will also have 30 minutes overtime, for a total of 3 and a half hours.

*1. Let $F = \mathbb{Q}(\sqrt{2}, i)$, and let $G := \mathrm{Gal}(F/\mathbb{Q})$ be the Galois group of F over $\mathbb{Q}$. Verify that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and explicitly list all the fields intermediate between F and $\mathbb{Q}$. Identify each intermediate field with its corresponding subgroup of G (as guaranteed by the Fundamental Theorem of Galois Theory).

2. $\mathbb{F}_{729}$ denotes the field with 729 elements. Note: $729 = 3^6$.

(a) What is the Frobenius map $\sigma_3 : \mathbb{F}_{729} \to \mathbb{F}_{729}$?

(b) Describe the elements of $\mathrm{Gal}(\mathbb{F}_{729}/\mathbb{F}_3)$, and the structure of this group.

(c) Draw a diagram of all the subfields of $\mathbb{F}_{729}$, and a diagram of all the subgroups of $\mathrm{Gal}(\mathbb{F}_{729}/\mathbb{F}_3)$.

3. For each of the following two cubics $f \in \mathbb{Q}[x]$, do the following four tasks:
(i) compute the discriminant of f;
(ii) state how many real roots f has; and
(iii) determine (up to isomorphism) both $\mathrm{Gal}(f/\mathbb{Q})$ and
(iv) $\mathrm{Gal}(f/\mathbb{R})$, the Galois groups of (the splitting fields of) f over $\mathbb{Q}$ and $\mathbb{R}$, respectively.
(If f is irreducible in $\mathbb{Q}[x]$, and if you use that fact, then explain why it's true.)

(a) $f(x) = x^3 - 4x + 10$

(b) $f(x) = x^3 - 21x + 7$

4. Suppose α is a real (or even a complex) number. Below, when we say that α is "constructible," we shall mean that it is constructible by means of straightedge and compass.

(a) State a necessary and sufficient condition for α to be constructible. No proof necessary.

(b) Show that $\cos 20°$ is not constructible. Here you may *not* use the fact that for every *integer* n, $\cos(n°)$ is constructible if and only if $3|n$. Instead, use the triple-angle formula $\cos\theta = 4\cos^3\theta/3 - 3\cos\theta/3$ (which you need not prove) and your answer to (a) above.

(This result shows that not every (constructible) angle can be trisected by straightedge and compass.)

5. Let F be a field, and let f be an irreducible polynomial in $F[x]$, with $\deg f = 6$. Let K be an extension field of F with $[K : F] = 2$. Prove that if f is reducible in $K[x]$, then it factors in $K[x]$ into the product of two irreducible cubic polynomials.

6. (a) Let K be a field. Prove that there are infinitely many monic, irreducible polynomials in $K[x]$.

(b) Show that every algebraically closed field K is infinite.