Instructions: *Please solve Problems 1 and 2 (the starred ones) and any two additional problems, for a total of four. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have three and a half hours.* Good luck!

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1. * Find the galois group G_f of the polynomial $f(X) = X^4 + 2$ over the following fields:

   (a) $\mathbf{Q}$

   (b) $\mathbf{Z}_3$,

   (c) $\mathbf{Z}_5$

2. * (a) Construct a field F having 125 elements (assume we know how to construct the splitting field of a polynomial).

   (b) Describe $Gal(F/F_5)$ (where F_5 is the field having five elements).

   (c) Describe the additive and the multiplicative groups of F in terms of cyclic groups, as assured by the Fundamental Theorem on Finite Abelian Groups.

3. (a) What does it mean the expression: "a finite field extension $F \subset E$ is normal"?

   (b) Prove: if the field extension $F \subset E$ satisfies $[E : F] = 2$, then it is normal.

   (c) Give an example of a finite, non-normal extension.

4. (a) Explain what it means the expression: " the finite group G is solvable".

   (b) Is every group of order 60 solvable? Briefly explain.

   (c) Let $f(X) \in \mathbf{Q}[X]$ be a polynomial whose Galois group G_f has order 46. Is $f(X)$ necessarily solvable by radicals? Explain.

5. Prove or disprove: if E is a subfield of $\mathbf{C}$ which is a Galois extension of $\mathbf{Q}$ and σ denotes complex conjugation, then $\sigma(E) = E$.

6. Let L be the splitting field of $f(X) = X^4 + X^2 + 1 \in \mathbf{Q}[X]$. Describe L and find $Gal(L/\mathbf{Q})$.

7. Show: if F is a field and $f \in F[X]$ is a polynomial of degree n with simple roots such that its Galois group G_f acts transitively on the set of roots of f, then f is irrreducible in $F[X]$.
