**Mathematics Comprehensive Examination**
**Algebra Core II (Field Theory and Galois Theory)**
**August 2004**

**Directions:** Do the first two problems, marked (*), and two additional problems for a total of four. Only turn in these four problems! Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet. You will be graded on the clarity as well as on the correctness of your response. You have three and one-half hours to complete this exam. Good luck!

1*. If $f(x) \in K[x]$ is irreducible of degree $d \geq 1$ and if $L|K$ is a finite extension of degree $n$ with $gcd(n, d) = 1$, then $f$ is irreducible in $L[x]$.

2*. a. Let $K$ and $L$ be fields and let $\phi : K \longrightarrow L$ be a map that is both additive and multiplicative:

$$\phi(a + b) = \phi(a) + \phi(b) \quad \text{and} \quad \phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

for all $a, b \in K$. Prove: If $\phi$ is not identically the 0-map, then $\phi(1_K) = 1_L$.

b. Let $L$ be a field containing the field of rational numbers. Prove that the only injective ring homomorphism from the rationals into $L$ is the identity map (= inclusion map of the rationals into $L$). (You may use without proof the fact that the rational number 1 equals the multiplicative identity of $L$.)

c. Prove that the only field automorphism of the field of real numbers is the identity.

3. Let $f(x)$ be a separable polynomial of degree $n$ that is irreducible in $K[x]$, where $K$ is a field. Let $\alpha$ be a root of $f(x)$ in a field $L$ containing $K$. Let $r$ denote the number of roots of $f(x)$ in $K[\alpha]$. Prove that $r$ divides $n$.

4. Derive a formula for the roots of the polynomial $x^3 + x + 1$. Show every step. (You are deriving the cubic formula from "first principles", so do **NOT** just write down the cubic formula.)

5. Prove or disprove: If $L|K$ is a finite galois extension and if $K|F$ is a finite galois extension, then $L|F$ is a finite galois extension.

6. Let $C_n$ denote a cyclic group of order $n$. Prove that there is a galois extension $K|Q$ whose Galois group is isomorphic to $C_n$. You may use Dirichlet's theorem stating that the arithmetic progression $1, 1 + n, 1 + 2n, 1 + 3n, 1 + 4n, \cdots$ contains infinitely many prime numbers.

7. Let $L|K$ be a finite galois extension and let $f(x) \in K[x]$ be an irreducible and separable polynomial of prime degree. Prove that $f(x)$ is either irreducible in $L[X]$ or that $f(x)$ factors into linear factors in $L[x]$.