CORE II – ALGEBRA

January 2003

Directions: Do exactly three of the following five problems. Do not hand in more than three problems. Please write your name clearly at the top of each sheet handed in. You have 3 and 1/2 hours to work. Good luck!

In the problems below, R and Q denote the fields of real numbers and rational numbers, respectively.

1. Prove that the only field automorphism of R is the identity. Show every step. You may use without proof the fact that a non-zero real number α is a square if and only if α is positive.

2. Let θ denote a primitive ninth root of unity.

a. Find the monic irreducible polynomial of θ over Q.

b. Find the monic irreducible polynomial of θ^3 over Q.

c. Let $N = Q[\theta]$. Give an explicit isomorphism between $Gal(N|Q)$ and the multiplicative group of units in the ring $Z/9Z$. Then use this isomorphism to prove that $Gal(N|Q)$ is cyclic.

3. Find explicit formulae for the roots (using square roots, cube roots, and rational numbers) of the cubic polynomial $X^3 + 3X^2 - 7X + 2$.

4. Let K be a field, and let $f(X)$ be a polynomial with coefficients in K. Prove that $f(X)$ has multiple roots (in some extension field of K) if and only if $f(X)$ and its formal derivative $f'(X)$ have a common root (in some extension field of K).

5. If $f(X) \in K[X]$ is irreducible of degree $d \geq 1$ and if $L|K$ is a finite extension of degree n with $gcd(n, d) = 1$, then $f(X)$ is irreducible in $L[X]$.