Mathematics Comprehensive Examination
Algebra Core II (Field Theory and Galois Theory)
January 2005

Directions: Do the first two problems, marked (*), and two additional problems for a total of four. Only turn in these four problems! Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet. You will be graded on the **clarity** as well as on the **correctness** of your response. You have three and one-half hours to complete this exam. Good luck!

1*. If $f(x) \in K[x]$ is irreducible of degree $d \geq 1$
and if $L|K$ is a finite extension of degree n with $\gcd(n, d) = 1$,
then f is irreducible in $L[x]$.

2*. Prove or disprove: If $L|K$ is a finite galois extension and if $K|F$ is a finite galois extension, then $L|F$ is a finite galois extension.

3. Let $f(x)$ be a separable polynomial of degree n that is irreducible in $K[x]$, where K is a field. Let α be a root of $f(x)$ in a field L containing K.
Let r denote the number of roots of $f(x)$ in $K[\alpha]$.
Prove that r divides n.
Hint: Where are the remaining $n - r$ roots of $f(x)$ not in $K[\alpha]$?

4. Give an explicit generator for each subfield of the splitting field of $x^4 + x^3 + x^2 + x + 1$ over Q.

5. Prove or disprove: If $[K : Q] = 3$ and if $[L : Q] = 3$ and if
$L \cap K = Q$ then the degree of $L \cdot K$ over Q is 9. Here, $L \cdot K$ denotes the compositum of L and K, that is, the smallest subfield of $\overline{Q}$
containing both L and K.

6. Let $L|K$ be a finite galois extension and let $f(x) \in K[x]$ be an irreducible and separable polynomial of prime degree. Prove that either $f(x)$ remains irreducible in $L[X]$ or that $f(x)$ factors completely in $L[x]$.