Mathematics Comprehensive Examination
Algebra Core II (Field Theory and Galois Theory)
January 2006

Do the starred problem (#1) and any three additional problems, for a total of four. The test is three and one-half hours long. Good Luck!

1.* Suppose $f \in \mathbb{Z}[x]$ is an irreducible quartic whose splitting field has Galois group S_4 over $\mathbb{Q}$. (There are many such quartics, but you need not prove this.) Let θ be a root of f, and set $K = \mathbb{Q}(\theta)$. Prove that K is an extension of $\mathbb{Q}$ of degree 4 that has no proper subfields (i.e., no intermediate fields other than K and $\mathbb{Q}$). (Hint: Use the Fundamental Theorem of Galois Theory.)

2. (a) Construct a field with precisely 27 elements.

(b) Classify the additive and multiplicative groups of this field (according to the Fundamental Theorem of Finite Abelian Groups).

(c) Determine the Galois group of this field over the field with 3 elements.

3. (a) Let F be a field and let $f \in F[x]$ be a polynomial. Show that f has multiple roots (in some extension field of F) if and only if f and its derivative f' have a common root.

(b) Give an example of an *irreducible* f having a multiple root.

4. Let p be a prime greater than 2. Show that if a regular p-gon is constructible with straightedge and compass, then $p = 1 + 2^n$, for some $n \in \mathbb{N}$.
(You need not show that in this case, n itself must also be a power of 2.)

5. For each of the following cubics $f \in \mathbb{Q}[x]$, compute the discriminant of f; determine the Galois group of (the splitting field of) f over $\mathbb{Q}$; and state how many real roots f has. (If f is irreducible, and if you use that fact, then explain why it's true.)

(a) $f(x) = x^3 - 3x + 1$

(b) $f(x) = x^3 - 4x + 2$

6. Let p be a prime, and $n \in \mathbb{N}$; write $q = p^n$. Show that for every (positive) divisor d of n, the finite field $\mathbb{F}_q$ has exactly one subfield of order p^d. Show also that for every $d \in \mathbb{N}$ not dividing n, $\mathbb{F}_q$ has no subfield of order p^d.

7. Suppose that G is a finite group, $H \triangleleft G$, and both H and G/H are solvable. Prove that G is solvable.