Mathematics Comprehensive Examination
Algebra Core II (Field Theory and Galois Theory)

January 2007

Do problem 1 and any three additional problems, for a total of four. You have a total of 3 and a half hours. Good luck!

1.* For each of the following three cubics $f \in \mathbb{Q}[x]$, do the following four tasks:
(i) compute the discriminant of f;
(ii) state how many real roots f has; and
(iii) determine (up to isomorphism) both $\mathrm{Gal}(f/\mathbb{Q})$ and
(iv) $\mathrm{Gal}(f/\mathbb{R})$, the Galois groups of (the splitting fields of) f over $\mathbb{Q}$ and $\mathbb{R}$, respectively.
(If f is irreducible in $\mathbb{Q}[x]$, and if you use that fact, then explain why it's true.)

(a) $f(x) = x^3 - 7x - 7$

(b) $f(x) = x^3 - 7x - 6$ (careful!)

(c) $f(x) = x^3 + 4x + 2$

2. (a) Construct a field K with precisely 16 elements.

(b) Does your field K in (a) contain a subfield with precisely 8 elements? If so, tell what it is; if not, tell why not.

(c) Determine $\mathrm{Gal}(K/\mathbb{F}_2)$, where $\mathbb{F}_2$ denotes the field with 2 elements.

3. Let K be a Galois extension of $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ is a cyclic group of order 28.

(a) How many intermediate fields are there, and what are their degrees over $\mathbb{Q}$?

(b) Give an example of such an extension K. (Hint: 29 is a prime.)

4. (a) Let K be a field. Prove that there are infinitely many monic, irreducible polynomials in $K[x]$. (*Hint*: Think of the integers.)

(b) Show that every algebraically closed field K is infinite.

5. Suppose that G is a finite, solvable group, and H is a subgroup. Prove that H is solvable.

6. (a) Show that an angle of $3°$ is constructible with straightedge and compass. Here you may use without proof the fact that a regular p-gon is constructible with straightedge and compass for any Fermat prime p. (*Hint*: Choose a suitable p.)

(b) Show that the angles $1°$ and $2°$ are not constructible with straightedge and compass. Here you may use without proof the fact that the angle $60°$ cannot be trisected with straightedge and compass.

(c) Prove that for every positive integer n, the angle $n°$ is constructible with straightedge and compass if and only if $3|n$. Here you may use the results of (a) and (b) above, even if you were not able to prove those results.

In (a), (b), and (c) above, you need not give detailed instructions for any constructions you use; instead, you may rely on well known facts about the constructibility of certain angles from other angles (any well known facts, that is, other than those in (a)–(c), of course).