Instructions: *Please solve Problems 1 and 2 (the starred ones) and any two additional problems, for a total of four. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have three and a half hours.* Good luck!

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1. * Show that the Galois group of $X^3 - 2 \in \mathbb{Q}[X]$ is isomorphic to the symmetric group S_3.

2. * Let $f(X) = X^p - X - a \in F[X]$, where F is a field of characteristic $p > 0$, E (a field extension of F) a splitting field of f over F. Show:

   (a) If $b \in E$ is a root of f, then $b, b+1, \ldots, b+p-1$ are all the roots of f.

   (b) If f is irreducible, then the Galois group $Gal(E/F)$ is cyclic, of order p

3. (a) Define "perfect field".

   (b) Show that a finite field is perfect.

   (c) Give an example of a field which is not perfect (show that your example works).

4. (a) Find the Galois group of the polynomial $x^8 - 1 \in \mathbb{Q}[X]$.

   (b) Find the Galois group of the polynomial $x^8 - 1 \in \mathbb{Z}_2[X]$.

5. (a) What does it mean "a field is algebraically closed"?

   (b) Prove that a finite field cannot be algebraically closed,

   (c) Prove that for any field F, the field $F[X]$ of rational functions (in the indeterminate X) over F cannot be algebraically closed.

6. Let $F \subset E$ be a field extension, a, b elements of E of degrees m and n over F, respectively, where m and n are relatively prime. Prove that $F[a, b]$ has degree mn over F (i.e., $[F[a, b], F] = mn$). *You may assume the fact that if $F \subset E \subset L$ are finite field extensions then $[L : F] = [L : E][E : F]$.*

7. Prove or disprove: the only automorphisms of the field $\mathbb{C}$ (of complex numbers) are the identity and complex conjugation.