## Core Exam II Graph Theory Fall 2003

**Instructions:** Solve any six from among the following seven problems. Submit only the six selected problems. You have 3 and 1/2 hours to complete this test. Good luck!

Graphs are finite and undirected, but may have loops and multiple edges unless stated otherwise.

1. Suppose G is a 2-connected graph that is not  $K_3$ , and e is an edge of G. Show that at least one of  $G \setminus e$  and G/e is also 2-connected.

2. Is there a planar 4-regular graph whose girth is four? Explain your answer.

3. Suppose G is a k-regular bipartite graph for some positive integer k. Prove that G has a perfect matching and use it to show that the edge chromatic number of G is k.

4. Prove (without invoking the Four Color Theorem) that if G is a planar Hamiltonian graph, then the dual of G is 4-colorable.

5. Suppose G is a connected loopless graph. Prove that  $\chi(G) \leq \Delta(G) + 1$ . Can  $\Delta(G)$  in the above inequality be replaced by a(G), where a(G) is the average vertex degree of G?

6. State the König-Egerváry Theorem and the Hall's Marriage Theorem. Use the former to prove the latter.

7. Prove that a graph is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of V(H).