Core-II Exam
Graph Theory
Spring, 2005

Instructions: *Solve any* **five** *from among the following seven problems. Submit* **only** *the five selected problems.* You have 3 and 1/2 hours to complete this test. Good luck!

Note: Graphs are finite, undirected, and have no loops and no multiple edges.

1. Prove or disprove: Every 3-regular graph has a perfect matching.

2. Let G be a 3-connected graph and let xy be an edge of G. Prove that G/xy is 3-connected if and only if $G - \{x, y\}$ is 2-connected.

3. Suppose the cubic graph G has exactly one edge-coloring with $\chi'(G)$ colors up to a permutation of the colors. Show that $\chi'(G) = 3$ and that G has exactly three Hamilton cycles.

4. Suppose A is a five-element set. Let the vertex set of G consist of all two-element subsets of A with two such sets being adjacent in G if and only if they are disjoint. Is G planar?

5. Describe all 2-connected graphs that have no even cycles.

6. Prove that every planar graph with at least four vertices has at least four vertices of degree less than six.

7. Let x be a vertex of a connected graph G and, for $r \geq 0$, let G_r be the subgraph of G induced by the vertices of distance r from x. Prove that $\chi(G) \leq \chi(G_r) + \chi(G_{r+1})$ for some r.