Instructions: *Do any 7 of the following 10 problems. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have three and a half hours.* Good luck!

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1. Find $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_1\cup\gamma_2}\frac{\zeta^3+3\zeta-4}{\zeta(\zeta-2i)(\zeta+4)}d\zeta$, where γ_1 is the curve $|z|=1$ oriented clockwise and γ_2 is the curve $|z-1|=4$ oriented counterclockwise.

2. Find the Laurent series for $\dfrac{z}{z^2-2z-3}$ in the annulus $2<|z|<3$.

3. Suppose $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ has a complex derivative $f'(z_0)$ at $z_0=x_0+iy_0$. Show that the Cauchy-Riemann equations for u and v are satisfied at z_0.

4. Answer the following questions. Give short justifications for your answers.
   (i) Does there exist a holomorphic mapping from $\mathbb{C}$ onto $D(0,1)$?
   (ii) Does there exist a holomorphic function from $D(0,1)$ onto $\mathbb{C}$? a biholomorphic one?
   (iii) Does there exist a biholomorphic map from $\mathbb{C}\setminus(-\infty,0]$, the complement of the nonpositive reals, to $D(0,1)$.
   (iv) Does there exist a biholomorphic map from an open annulus to the open unit disk?
   (v) Does every holomorphic function defined on $\mathbb{C}\setminus(-\infty,0]$ have a holomorphic antiderivative?
   (vi) Does every holomorphic function on $\mathbb{C}\setminus\{0\}$ have a holomorphic antiderivative?
   (vii) Does every holomorphic function on $\mathbb{C}\setminus\{0\}$ have a holomorphic derivative?

5. Compute the following; show work:
   (i) $\displaystyle\int_\gamma \frac{dz}{z^2}$, $\quad\gamma(t)=\cos t+i\sin t$, $0\le t\le\frac{\pi}{2}$
   (ii) $\displaystyle\oint_{|z|=2}\overline{z}\,dz$, oriented counterclockwise
   (iii) $\displaystyle\oint_{|z-1|=2}e^{z^2}dz$, clockwise orientation
   (iv) $\displaystyle\oint_{|z|=2}\frac{\sin^2 z}{(z-\frac{\pi}{4})^2}dz$, oriented counterclockwise.

6. Compute using contour integration $\displaystyle\int_0^\infty\frac{1}{1+x^4}dx$; show work.

7. (i) Use the Cauchy estimate

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right| \leq \frac{Mk!}{r^k} \quad \text{where} \quad \sup\{|f(z)| : |z-P| = r\} \leq M$$

to characterize those entire functions f that satisfy for some constant K, $|f(z)| \leq K|z|$ for all $|z| \geq 1$.
(ii) Use (i) to characterize those functions f that are holomorphic on $\mathbb{C} \setminus \{0\}$, have a simple pole at 0, and are bounded on the complement of $D(0,1)$.

8. Let $f(z) = z^8 + 2z^4 + 8z - 2$. Determine the number of zeros of f, appropriately taking into count multiplicity, in each of the following sets. Justify your conclusions.
(i) $|z| < 1$
(ii) $|z| \geq 2$
(iii) $|z| = 1$
(iv) $1 < |z| < 2$.

9. Let γ be a closed curve whose image is contained in an open set U. Suppose that for any holomorphic function $F(z)$ defined on U, $\int_\gamma F(z)dz = 0$. By considering the difference quotient funcition, prove for any holomorphic map f on U and any z not in the image of γ, Cauchy's Integral Formula

$$\operatorname{Ind}_\gamma(z) \cdot f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

is valid.

10. Let $\mathcal{F}$ be the family of all holomorphic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$$

on $|z| < 1$, where $|a_n| \leq n$ for all $n \geq 2$. Show that $\mathcal{F}$ is a normal family.