**Instructions**: *Do any 7 of the following 10 problems. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have three and a half hours.* Good luck!

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1. Let $f(z) = \sum_{n=1}^{\infty} z^n/n^2$.
   (i) Find the radius of convergence $R$ of the power series defining $f$.
   (ii) Argue that the series converges absolutely and uniformly on the boundary $|z| = R$.
   (iii) Is it possible to extend $f$ so that it is analytic on some open disk $D(0, r)$ for some $r > R$? Explain.
   (iv) Give the Taylor series for $f'(z)$.

2. Find $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{e^\zeta}{\zeta^3 + 3\zeta^2 + 2\zeta}d\zeta$, where $\gamma$ is the triangle with vertices $-3, 1 \pm i$ oriented counterclockwise.

3. Find the Laurent series around 0 for $\displaystyle\frac{z-3}{z^2+3z}$ and the annulus on which it converges.

4. Prove that there is no holomophic function $f$ that is holomorphic on $\mathbb{C} \setminus \{0\}$ and has derivative $f'(z) = 1/z$ for $z \neq 0$.

5. For $f = u + iv : U \to \mathbb{C}$, where $u, v : U \to \mathbb{R}$ are $C^1$, we define
   $$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
   and
   $$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$
   (i) Find $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ for $f(z) = z$ and for $f(z) = \overline{z}$.
   (ii) Prove that
   $$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}}, \text{ where } \overline{f} = u - iv.$$
   (iii) One (equivalent) definition for a holomorphic function on some domain $U$ is that $\partial f/\partial \overline{z} = 0$ on $U$. Show that $f$ is holomorphic on $U$ if and only if $\partial \overline{f}/\partial z = 0$ on $U$.

6. Compute using contour integration $\displaystyle\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$; show work.

7. (i) Use the Cauchy estimate

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right| \leq \frac{Mk!}{r^k} \quad \text{where} \quad \sup\{|f(z)| : |z-P| = r\} \leq M$$

to characterize those entire functions $f$ that satisfy for some constant $K$, $|f(z)| \leq K|z|$ for all $|z| \geq 1$.
(ii) Use (i) to characterize those functions $f$ that are holomorphic on $\mathbb{C} \setminus \{0\}$, have a simple pole at 0, and are bounded on the complement of $D(0,1)$.

8. Let $f(z) = z^6 + z^3 + 5z^2 - 2$. Determine the number of zeros of $f$, appropriately taking into count multiplicity, on $1 < |z| < 2$. Justify your conclusion.

9. Let $U$ be an open set, let $P \in U$, and let $f : U \setminus \{P\}$ be holomorphic. Let $s(z)$ be the principal part (=negative terms) of the Laurent series expanded around $P$ and evaluated at $z$. Set $g(z) = f(z) - s(z)$ for all $z$ where the right hand exists. Argue that $g$ extends to a holomorphic function on all of $U$.

10. Let $\sum_{n=0}^{\infty} c_n$ be a series of positive terms such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. Let $\mathcal{F}$ be the family of all holomorphic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$, where $|a_n| \leq c_n$ for all $n \geq 0$.
(i) Show that the radius of convergence for each member of $\mathcal{F}$ is $\infty$, so that the power series defines an entire function.
(ii) Show that $\mathcal{F}$ is a normal family.