CORE II **DIFFERENTIAL GEOMETRY EXAM** **SUMMER 2005**

Instructions. Do four of the following five problems. You have three and a half hours.

1. Let M be a smooth manifold and let $\widetilde{M}$ be a topological manifold. Suppose $\pi : \widetilde{M} \to M$ is a surjective continuous map with the property that every $p \in M$ has an open neighborhood U such that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U. Show that $\widetilde{M}$ has a unique smooth structure with respect to which π is a smooth map.

2. Let M be a topological n-manifold and let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be a smooth atlas for M. Suppose that for each $\alpha, \beta \in A$ there is a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$. Suppose further that for all $\alpha, \beta, \gamma \in A$:
$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \quad p \in U_\alpha \cap U_\beta \cap U_\gamma \ .$$
Let T be the disjoint union of the sets $U_\alpha \times \mathbb{R}^k$, $\alpha \in A$. Define in T a relation $\sim$ by setting $(p, v) \sim (p', v')$ if and only if $(p, v) \in U_\alpha \times \mathbb{R}^k$, $(p', v') \in U_\beta \times \mathbb{R}^k$, $p = p'$ and $\tau_{\alpha\beta}(p)(v) = v'$.
   a. (i) Show that $\sim$ is an equivalence relation on T. (ii) Let $E := T/\sim$ be the topological quotient space; *i.e.*, $X \subset E$ is open iff $\bigcup X$ is open in T. (Recall that the elements of E are equivalence classes in T.) Let $f_\alpha : U_\alpha \times \mathbb{R}^k \to E$ be the natural map. Show that f_α is a homeomorphism onto its image.
   b. Let W_α be the image of f_α and let $\Phi_\alpha := {f_\alpha}^{-1}$. Show that $\{(W_\alpha, \Phi_\alpha) \mid \alpha \in A\}$ is a smooth atlas on E. (Here, to avoid excessive notation, you are expected to identify U_α with $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.)
   c. Show that there is a well-defined *smooth* surjection $\pi : E \to M$ such that $\pi^{-1}(p) = \mathbb{R}^k$ for each $p \in M$, $\pi^{-1}(U_\alpha) = W_\alpha$ and $\pi|_{W_\alpha} = \pi_\alpha \circ \Phi_\alpha$, where $\pi_\alpha : U_\alpha \times \mathbb{R}^k \to U_\alpha$ is the projection; conclude that $\pi : E \to M$ is a smooth vector bundle.

3. Suppose $\Phi : M \to N$ is a smooth map and $S \subset N$ is an embedded submanifold. Assume that for every $p \in \Phi^{-1}(S)$ the spaces $T_{\Phi(p)}S$ and $\Phi_* T_p M$ together span $T_{\Phi(p)}N$ (*i.e.*, Φ is *transverse* to S). Show that $\Phi^{-1}(S)$ is an embedded submanifold of M whose codimension is equal to $\dim N - \dim S$. (You may use the following facts: 1) every regular level set of a smooth map is a closed embedded submanifold whose codimension is equal to the dimension of the range; 2) a subset of a smooth n-manifold is an embedded k-submanifold iff locally it is the level set of a submersion in $\mathbb{R}^{n-k}$.)

4. Show that any two points in a connected smooth manifold can be joined by a smooth curve segment. (If you use the Whitney Approximation Theorem, then state it or whatever part of it you use.)

5. Define a 2-form Ω on $\mathbb{R}^3$ by $\Omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$.
   a. Compute Ω is spherical coordinates (ρ, ϕ, θ) definded by
$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\theta).$$
   b. Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
   c. Compute the restriction $\Omega|_{S^2} = \iota^*\Omega$, using coordinates (ϕ, θ), on the open set where these coordinates are defined.
   d. Show that $\Omega|_{S^2}$ is nowhere zero.