Instructions: *Do any 8 of the following 12 problems. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have three and a half hours.* Good luck!

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1. Give the definition of an atlas and a differentiable structure for a manifold. Tell how to obtain a differentiable structure from an atlas and prove that there is only one differentiable structure containing a given atlas.

2. Define a smooth map and a diffeomorphism between manifolds. Prove that to establish smoothness it suffices to check smoothness on an open cover of chart domains, one for the domain manifold and one for the range.

3. Give the definition of an n-dimensional submanifold N of an m-dimensional manifold M, where $n < m$. Show that the graph of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + {x_3}^2 - 4$ is a submanifold of $\mathbb{R}^4$ by giving a suitable "slice chart" diffeomorphism $\psi : \mathbb{R}^4 \to \mathbb{R}^4$. Show that ψ is indeed a diffeomorphism by finding a smooth inverse and show that ψ satisfies the slice chart property.

4. State the "full-rank" submanifold theorem. Verify that it applies to show that the unit sphere S^n is a submanifold of $\mathbb{R}^{n+1}$.

5. Explain how in differential geometry $f_* = Tf : TM \to TN$ is defined, where $f : M \to N$ is a differentiable map between manifolds.
(i) What basic diagram involving f and Tf commutes?
(ii) What is Tf for f a constant map.
(iii) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map? Explain TA.
(iv) A function $F : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $F(x, y, z) = (x^2 + y^2 + z^2, xyz)$. Find a formula for $F_* : T\mathbb{R}^3 \to T\mathbb{R}^2$.

6. State the inverse function theorem. Use it to establish an inverse function theorem for smooth manifolds. How do you use this to show that a smooth bijection $f : M \to N$ between manifolds such that for every p, $f_*(p) : T_pM \to T_{f(p)}N$ is invertible is a diffeomorphism.

7. Very briefly indicate three different ways that one can define the tangent spaces T_pM at points p of a manifold M.

8. What is a smooth vector field on a manifold M? Define a local flow and explain how one obtains the local flow generated by a smooth vector field X. How does one recover the original vector field from the local flow?

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9. Find the vector field on $\mathbb{R}^3$ that generates the flow $\Phi : (t,(x,y,z)) = (x\cos t - y\sin t, x\sin t + y\cos t, z+t)$. Verify that the semigroup property $\Phi(t,\Phi(s,(x,y,z))) = \Phi(t+s,(x,y,z))$ is satisified for the given Φ. (Hint: This may involve some elementary trig identities.)

10. Give the definition of a smooth bundle over a manifold M and the definition of a bundle map between two bundles. Give the definition of a global frame. Show that a trivial bundle has a global frame. Does the converse hold?

11. (i) Define a derivation on the ring (or algebra) $C^\infty(M)$ of smooth functions from M into $\mathbb{R}$. Define the Lie product of two derivations and show that it is again a derivation.
(ii) Describe the equivalance between the notion of a smooth vector field on M and a derivation on $C^\infty(M)$. Your answer should include a description of how a smooth vector field gives rise to a derivation and vice-versa.

12. For a finite dimensional real vector space V, give the definition of linear functional and dual space, and tell given a basis of V, the dual basis is defined. Using these ideas, explain how the cotangent bundle is obtained.