Instructions: *Do any* 6 *of the following* 8 *problems. All spaces that arise on this exam are Hausdorff and second countable, and all manifolds are smooth. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have three and a half hours.* Good luck!

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1. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$, and define an equivalence relation on X by $x \sim y$ if $y = \lambda x$ for some nonzero scalar λ. Let $P^n(\mathbb{R}) = X/\sim$ be the corresponding quotient space. Show that $P^n(\mathbb{R})$ is a smooth manifold of dimension n.

2. Let $\psi : M \to N$ be a C^∞ mapping from the connected manifold M to the manifold N.

   (a) Let $p \in M$. Define $d\psi_p$, the differential of f at p.

   (b) If $d\psi_p \equiv 0$ for each $p \in M$, show that ψ is a constant map.

   (c) Use part (b) to show that if M is a connected manifold, then the de Rham cohomology group $H^0(M)$ is isomorphic to $\mathbb{R}$.

3. Let M be a d-dimensional manifold, and let $p \in M$.

   (a) Give a definition of M_p, the tangent space to M at p.

   (b) Give a definition of TM, the tangent bundle over M, and explain how one may construct coordinate charts for TM using coordinate charts for M.

   (c) If $f : M \to N$ is a C^∞ map, show that f induces a map $T(f) : TM \to TN$ on tangent bundles.

4. (a) Define the terms *immersion*, *submanifold*, *imbedding*, and *diffeomorphism*.

   (b) Give an example of a map $f : M \to N$ that is an immersion but is not a submanifold, and give an example of a map $g : M \to N$ that is one-to-one but is not an immersion.

   (c) State the implicit function theorem and use it to show that the natural inclusion of the n-sphere S^n in $\mathbb{R}^{n+1}$ is an imbedding.

5. Let $C^\infty(M)$ denote the algebra of smooth functions on the smooth manifold M.

   (a) Define a derivation on the algebra $C^\infty(M)$.

   (b) Define the Lie product of two derivations on $C^\infty(M)$, and show that it is again a derivation.

   (c) Describe the equivalence between smooth vector fields on M and derivations on $C^\infty(M)$. Your answer should include a description of how a smooth vector field gives rise to a derivation, and visa-versa.

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6. Let V be a real, finite dimensional vector space of dimension $\dim V = n$.

   (a) Define the exterior algebra ΛV, and exhibit a basis for ΛV.

   (b) Show that two linearly independent sets $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ in V span the same r-dimensional subspace if and only if $v_1 \wedge \cdots \wedge v_r = \det A \cdot w_1 \wedge \cdots \wedge w_r$, where $A = (a_{i,j})$ is given by $v_i = \sum_{j=1}^{r} a_{i,j} w_j$.

7. Let X be an n-dimensional smooth manifold, and $f : X \to \mathbb{R}^{n+1}$ an immersion. A normal vector field along (X, f) is a smooth map $N : X \to T(\mathbb{R}^{n+1})$ such that for each $p \in X$, the vector $N(p)$ lies in $(\mathbb{R}^{n+1})_{f(p)}$ and is orthogonal to the subspace $df(X_p)$ (with respect to the standard inner product).

   (a) Show that if there is a smooth nowhere-vanishing normal vector field along (X, f), then the manifold X is orientable.

   (b) Use part (a) to show that S^n, the n-dimensional sphere, is orientable.

8. Consider the one-form $\omega = \dfrac{x}{x^2+y^2}dy - \dfrac{y}{x^2+y^2}dx$ on $\mathbb{R}^2 \setminus \{0\}$.

   (a) Show that ω is closed.

   (b) Evaluate the integral of ω over the unit circle S^1. How does this result show that ω is not exact? What conclusion can be drawn regarding the de Rham cohomology group $H^1(\mathbb{R}^2 \setminus \{0\})$?

   (c) Let $i : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ denote the natural inclusion. Is the restriction $\delta i(\omega)$ exact? Explain.