Manifolds are assumed to be $C^\infty$ manifolds if not otherwise stated. Also all topological spaces including manifolds are assumed to be second countable and Hausdorff.

**You must do four of the the first five problems**. Then *select five of the remaining eleven to submit for a total of nine problems. You have 3 and 1/2 hours for this test.* Good luck!

1. (a) Give the definition of a $d$-dimensional $C^\infty$ atlas on Hausdorff topological space $M$.
   (b) Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two $d$-dimensional $C^\infty$ atlases on a manifold $M$. What does it mean to say these atlases are compatible; i.e., define the same manifold structure on $M$?
   (c) Give an atlas $\mathcal{A}$ on $\mathbb{R}$ which defines the standard manifold structure for $\mathbb{R}$.
   (d) Give an atlas $\mathcal{B}$ for $\mathbb{R}$ consisting of one element such that the manifold structure defined by $\mathcal{B}$ is not compatible with the manifold defined by $\mathcal{A}$. Moreover, find such a $\mathcal{B}$ such that every real valued $C^\infty$ function relative to $\mathcal{B}$ is $C^\infty$ relative to $\mathcal{A}$.
2. (a) State the $C^\infty$ inverse function theorem for a $C^\infty$ function $f : U \to \mathbb{R}^n$ where $U$ is an open subset of $\mathbb{R}^n$.
   (b) Use this theorem to show if $x_1, x_2, \dots, x_n$ are a $C^\infty$ coordinate system for a manifold $M$ on an open neighborhood $U$ of $p$, then $y_1, y_2, \dots, y_n$ defined by $(y_1, y_2, \dots, y_n) = f(x_1, x_2, \dots, x_n)$ is a coordinate system on some neighborhood of $p$ if and only if the Jacobian matrix $[D_i f_j(x_1(p), \dots, x_n(p))]$ is invertible. Here $f = (f_1, f_2, \dots, f_n)$.
3. Let $M$ be a $C^\infty$ $d$-manifold.
   (a) Let $p \in M$. Give the definition of a vector $X$ at $p$ as a derivation into $\mathbb{R}$.
   (b) Let $x_1, x_2, \dots, x_d$ be a coordinate system defined on an open neighborhood $U$ of $p$ in $M$. Define the vectors $\frac{\partial}{\partial x_i}(p)$ at the point $p$.
   (c) State the basis theorem for the vector $X$ at $p$.
4. Consider the circle $S^1$ in $\mathbb{R}^2$ consisting of all points $(a, b) \in \mathbb{R}$ with $a^2 + b^2 = 1$. Let $\mathcal{A}$ consist of the following charts:

$$\xi^+ : \{(a, b) \in S^1 \mid a > 0\} \to \mathbb{R} \text{ given by } \xi^+(a, b) = b$$
$$\xi^- : \{(a, b) \in S^1 \mid a < 0\} \to \mathbb{R} \text{ given by } \xi^-(a, b) = b$$
$$\eta^+ : \{(a, b) \in S^1 \mid b > 0\} \to \mathbb{R} \text{ given by } \eta^+(a, b) = a; \text{ and}$$
$$\eta^- : \{(a, b) \in S^1 \mid b < 0\} \to \mathbb{R} \text{ given by } \eta^-(a, b) = b.$$

   (a) Show these charts are $C^\infty$ differentiably related and thus make $S^1$ into a $C^\infty$ 1-manifold.
   (b) Show the function $\theta$ defined on $S^1$ by $\theta(\cos t, \sin t) = t$ where $-\pi < t < \pi$ defines a $C^\infty$ coordinate system on $S^1$.
   (c) Write the vector $\frac{d}{d\theta}|_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$ in terms of the vector $\frac{d}{d\xi^+}|_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$.
5. Let $f$ be a $C^\infty$ differentiable function from $C^\infty$ $d$-manifold $M$ into $C^\infty$ $e$-manifold $N$.
   (a) Let $p \in M$. Give the definition of the differential of $f$ at $p$; i.e., give the definition of $df_p$ (also called $f_{*p}$, $df(p)$.).
   (b) Suppose $M$ is connected. Show $f$ is constant if and only if $df_p = 0$ for all $p \in M$.

6. Let $M$ and $N$ be $C^\infty$ manifolds. Let $f : M \to N$.
   (a) Give a definition that $f$ is $C^\infty$ differentiable from $M$ into $N$.
   (b) Give the definition that $M$ is $C^\infty$ diffeomorphic to $N$.
   (c) Let $M = N = \mathbb{R}$ with the usual topology. Let $M$ have atlas $\{x\}$ where $x : \mathbb{R} \to \mathbb{R}$ is given by $x(a) = a$ and $N$ have atlas $\{c\}$ where $c(a) = a^3$. Show $M$ and $N$ are diffeomorphic.
7. Let $M = \mathbb{R}^d$ and $N = \mathbb{R}^e$ with their standard manifold structures. Let $p \in M$. Show the differential $dT_p$ of a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^e$ relative to the standard bases of the tangent spaces $M_p$ and $N_{T(p)}$ is the same as the matrix of $T$ relative to the usual vector space bases of $\mathbb{R}^d$ and $\mathbb{R}^e$.
8. (a) Give the definition of a vector field $X$ on a $C^\infty$ $d$-manifold $M$.
   (b) Using the definition that $X$ is $C^\infty$ if and only if $Xf$ is a $C^\infty$ real valued function whenever $f$ is a real valued $C^\infty$ function on $M$, show that the Lie bracket $[X, Y]$ of two $C^\infty$ vector fields $X$ and $Y$ is again a $C^\infty$ vector field.
9. Consider the vector fields $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$, and $R = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $\mathbb{R}^2$.
   (a) Show their linear span is a Lie algebra.
   (b) Find the integral curve of $aX + bY + cR$ starting at point $(1, 0)$ in $\mathbb{R}^2$.
10. Let $M = \mathbb{R}^3$ and let $X$ be the $C^\infty$ vector field given by $X_{(a,b,c)} = \langle -bc, ac, c\rangle = -bc\frac{\partial}{\partial x}(a, b, c) + ac\frac{\partial}{\partial y}(a, b, c) + c\frac{\partial}{\partial z}(a, b, c)$ or more succinctly by $X = -yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Let $\phi$ be an integral curve for $X$. Set $\phi_1 = x \circ \phi$, $\phi_2 = y \circ \phi$, $\phi_3 = z \circ \phi$. Determine the first order system of differential equations that these three functions satisfy.
11. Recall if $M$ is a $d$-manifold with atlas $\mathcal{A}$ and $N$ is a $e$-manifold with atlas $\mathcal{B}$, then $M \times N$ with product topology is a $d+e$ manifold with atlas $\mathcal{A}\times\mathcal{B} = \{\xi\times\eta \mid \xi \in \mathcal{A},\ \eta \in \mathcal{B}\}$ where $\xi\times\eta : \mathrm{Dom}(\xi)\times\mathrm{Dom}(\eta) \to \mathbb{R}^{d+e}$ is given by $\xi \times \eta(p, q) = (\xi(p), \eta(q))$. Let $P$ be a $k$-manifold. Show a function $\phi = (\phi_1, \phi_2)$ mapping $P$ into $M \times N$ is differentiable if and only if $\phi_1 : P \to M$ and $\phi_2 : P \to N$ are differentiable.
12. Let $M$ be a $e$-manifold. Let $S$ be a subset of $M$ which is also a $d$-manifold.
    (a) Give the definition of $S$ is a submanifold of $M$.
    (b) Show if $S$ is a submanifold of $M$, then the topology of $S$ is stronger or equal to the relative topology of $M$ on $S$.
    (c) Give an example where the topology on $S$ is strictly stronger than the relative topology from $M$.
13. Let $I$ and $J$ be nonempty open intervals in $\mathbb{R}$. Suppose $\alpha : I \to M$ is an integral curve for a nonvanishing $C^\infty$ vector field $X$ on a $d$-manifold $M$ and $h : J \to I$ is a differentiable mapping. Show $\alpha \circ h : J \to M$ is an integral curve for $X$ if and only if $h(t) = t + c$ for some constant $c$.
14. Let $M$ be a $d$-manifold.
    (a) Give the definition of a $d + k$ $C^\infty$ vector bundle $E$ over $M$.
    (b) Let $T^*(M)$ be the cotangent bundle; i.e., the space consisting of all pairs $(p, \omega)$ where $p \in M$ and $\omega : M_p \to \mathbb{R}$ is linear. Give an atlas for this manifold.
    (c) Show $T^*(M)$ is a $2d$-vector bundle over $M$.
15. Let $M$ be a $d$-manifold.
    (a) Give the definition of a Riemannian metric on $M$.
    (b) Let $S$ be a submanifold of a Riemannian manifold $M$. Let $j$ be the inclusion mapping from $S$ into $M$. Show $S$ has a unique Riemannian metric such that for each $p \in S$, the differential $dj_p : S_p \to M_p$ is an isometry.
    (c) Suppose now $M$ and $S$ are connected. Define the Riemannian distance between two points in $M$; and similarly do so for $S$. Then show the $S$ Riemannian distance between two points in $S$ is greater or equal to the $M$ Riemannian distance between the same two points.