Fall, 2003 **Differential Geometry 7550**

Comprehensive Examination

Instructions: Do any 10 of the following 15 problems. *You have 3 and 1/2 hours.* Good Luck!

1. Give the definition of an atlas and a differentiable structure. Tell how to obtain a differentiable structure from an atlas. Prove that an atlas uniquely determines a differentiable structure.

2. Give the definition of an n-dimensional submanifold N of an m-dimensional manifold M. Show that the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 4x^2 + y^2 - 4$ is a submanifold of $\mathbb{R}^3$ by giving a suitable "slice chart."

3. State and prove the "full-rank" submanifold theorem. Verify that it applies to show that the unit sphere S^2 is a submanifold of $\mathbb{R}^3$.

4. A function F from $\mathbb{R}^3$ to $\mathbb{R}^2$ is defined by the rule $F(x, y, z) = (x^2 + y^2, xy/z)$.
(i) What open set U is the domain of F, the largest set on which F is defined?
(ii) Compute $F_* : TU = U \times \mathbb{R}^3 \to T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$.
(iii) What is $F_*((-1, 2, 1), (2, 1, 2))$?

5. Find the global flow $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ generated by the vector field $X(x, y, z) = ((x, y, z), (x + 1, 2, -z))$.

6. What is an immersion? State the immersed submanifold theorem.

7. A function F from $\mathbb{R}^3$ to $\mathbb{R}^2$ is defined by the rule $F(x, y, z) = (x^2 + y^2, xy/z)$.
(i) What open set U is the domain of F, the largest set on which F is defined?
(ii) Compute $F_* : TU = U \times \mathbb{R}^3 \to T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$.
(iii) What is $F_*((-1, 2, 1), (2, 1, 2))$?

8. What is a partition of unity on a smooth manifold M. What does it mean to say that it is subordinate to some given open cover $\mathcal{U}$. What is the existence theorem for partitions of unity?

9. Briefly indicate three different ways that one can define the tangent spaces T_pM at points p of a manifold M.

10. Let $M = \mathbb{R}^2$. The correspondence between vector fields and derivations associates with the vector field $X(x, y) = ((x, y), (-y, x))$ the derivation $Xf = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$.
(i) If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = x^2 + 2xy + y^2$, find the function Xf.
(ii) Verify that X is a derivation.

11. What is a smooth vector field on a manifold M? Define a local flow and explain how one obtains the local flow generated by a smooth vector field X. How does one recover the original vector field from the local flow?

12. Find the vector field on $\mathbb{R}^3$ that generates the flow $\Phi : (t, (x, y, z)) = (x \cos t - y \sin t, x \sin t + y \cos t, z + t)$. Verify that the semigroup property $\Phi(t, \Phi(s, (x, y, z))) = \Phi(t + s, (x, y, z))$ is satisified for the given Φ. (Hint: This may involve some elementary trig identities.)

13. (i) Give the definitions of a smooth bundle over a manifold M and of a of a bundle map between two bundles.

(ii) Define the Möbius line bundle over the circle by defining the trivializing neighborhoods, the trivializing maps, and checking the compatibility conditions.

14. Compute using the machinery of differential geometry the area of the parametrized cylinder $\phi(\theta, t) = (2\cos\theta, 2\sin\theta, t)$, $0 < \theta < 2\pi$, $0 < t < 3$.

15. (i) Define a derivation on the ring (or algebra) $C^\infty(M)$ of smooth functions from M into $\mathbb{R}$. Define the Lie product of two derivations and show that it is again a derivation.

(ii) Describe the equivalance between the notion of a smooth vector field on M and a derivation on $C^\infty(M)$. Your answer should include a description of how an smooth vector field gives rise to a derivation and vice-versa.