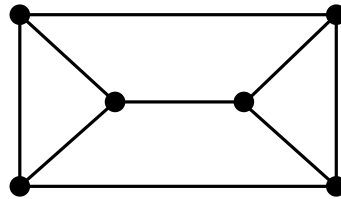


**Core II Exam**  
**Graph Theory**  
August 2005

**Instructions:** Solve any five from among the following seven problems. Submit only the five selected problems. You have 3 and 1/2 hours to complete this test. Good luck!

*Note: Graphs are finite, undirected, and have no loops and no multiple edges.*

1. Use Brooks' Theorem to give a short proof of a special case of Vizing's Theorem: If  $\Delta(G) = 3$ , then  $\chi'(G) \leq 4$ .
2. Without invoking any major theorems on perfect graphs, prove that the complement of a path is perfect.
3. Prove that  $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{|G|}$ .
4. Derive Hall's Marriage Theorem from Tutte's 1-Factor Theorem.
5. Find the smallest number  $k$  such that the Prism Graph, which is depicted below, admits a nowhere-zero  $k$ -flow. Give an example of such a  $k$ -flow.



The Prism Graph

6. Prove that every planar graph is a union of three forests.
7. Let  $G$  be a plane triangulation, and let  $n_i$  be the number of vertices of degree  $i$  in  $G$ . Prove that

$$\sum (6 - i)n_i = 12.$$