## Graph Theory Core II Comprehensive Exam

Solve any six from among the following eight problems. Explain all of your answers. Submit only the six selected problems. You have 3 and 1/2 hours to complete this test. Good luck!

Note: Graphs are finite, undirected, and have no loops and no multiple edges.

- 1. Let G is a bipartite graph with bipartition (U, W), where |U| = |W| = n, and minimum degree at least n/2. Show that G has a perfect matching.
- 2. Show that every cubic Hamiltonian graph has at least three different Hamilton cycles.
- 3. Prove or disprove: Every graph with no subgraph isomorphic to  $K_4$  has chromatic number at most 3.
- 4. Consider a drawing of a regular pentagon with all possible diagonals drawn, to give the usual depiction of  $K_5$ , but view it as a rotation system of  $K_5$  embedded in some surface. What surface is that?
- 5. Determine the flow number of  $W_5$ , the wheel with five spokes, without invoking the flow-coloring duality theorem.
- 6. What is the chromatic number of the dodecahedron?
- 7. Suppose G is a perfect graph and e is an edge of G. Is it necessarily true that  $G \setminus e$  is also perfect? What about G/e?
- 8. Let G be a 3-connected graph, and xy be an edge of G. Show that G/xy is 3-connected if and only if  $G \{x, y\}$  is 2-connected.