Core 2 Exam: Measure and Integration
January 2004

Instructions: Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours. Good luck!

1. Let μ be a Borel measure on $\mathbf{R}^N$ with $\mu(\mathbf{R}^N) = 1$. Let $f \in L^1(\mathbf{R}^N)$. Define

$$\hat{f}(t) = \int_{\mathbf{R}^N} e^{i\langle t,x\rangle} f(x)\, dx, \qquad \text{and} \qquad \hat{\mu}(x) = \int_{\mathbf{R}^N} e^{i\langle t,x\rangle}\, d\mu(t)$$

for every $t \in \mathbf{R}^N$. Here $\langle t, x\rangle$ is the usual inner–product of $t, x \in \mathbf{R}^N$.
(i) Prove that $\hat{\mu}$ is a continuous function
(ii) Prove the identity

$$\int_{\mathbf{R}^N} \hat{f}\, d\mu = \int_{\mathbf{R}^N} f(x)\hat{\mu}(x)\, dx,$$

clearly justifying all steps.

2. (i) Let f be a non-negative measurable function on a measure space $(X, \mathcal{B}, \mu)$, with $\int f\, d\mu = 0$. Prove that f is μ–almost everywhere equal to 0.
(ii) Deduce, with explanations, the value of the limit

$$\lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbf{R}} f(x) e^{-x^2/(2\sigma^2)}\, dx,$$

where f is a bounded, measurable function on $\mathbf{R}$ which is continous at 0.

3. Let H be a Hilbert space, M a closed subspace of H, and $x \in H$. Prove that there is a unique point $y \in M$ which is closest to x.

4. Let $p, q \in (1, \infty)$ be conjugate indices, and $(X, \mathcal{F}, \mu)$ a measure space. Let $f \in L^p(\mu)$. Find a function $g \in L^q(\mu)$ with $\|g\|_q \le 1$ such that

$$\int fg\, d\mu = \|f\|_p$$

5. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\mathcal{F}'$ be the collection of all subsets E of X with the property that there exist $A, B \in \mathcal{F}$ and $A \subset E \subset B$ with $\mu(B - A) = 0$. Show that $\mathcal{F}'$ is a σ–algebra.

6. Let μ be a finite Borel measure on a locally compact Hausdorff space in which every open set is a countable union of compact sets. Prove that every Borel set is regular with respect to μ, i.e. if E is any Borel set and $\epsilon > 0$ then there is a compact set K and an open set U with $K \subset E \subset U$ and $\mu(U - K) < \epsilon$.

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7. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $1 \leq p < \infty$. Prove the triangle inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for all $f, g \in L^p(\mu)$. (You may use the result of Problem 4 if you wish, along with any other standard L^p–L^q inequalities.)

8. Let V be a complex vector space and $g : V \to \mathbf{R}$ a real–linear functional. Show that there is a complex–linear functional $f : V \to \mathbf{C}$ such that g is the real part of f. Now suppose the complex vector space is equipped with a norm with respect to which g is continuous. Prove that f is also continuous.