Core 2 Exam: Measure and Integration
August 2002

Instructions: *Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours. Good luck!*

1. Calculate the volume of a ball of radius R in $\mathbf{R}^n$ by any reasonable method.

2. Let f be a measurable function on $\mathbf{R}$. In the following, the measure on $\mathbf{R}$ is Lebesgue measure.

   (i) Suppose $f \in L^1(\mathbf{R})$. For any real number $k \in \mathbf{R}$, define

$$\hat{f}(k) = \int_{\mathbf{R}} e^{ikx} f(x)\, dx$$

   Prove that $\hat{f}$ is a continuous function.

   (ii) Suppose f is a measurable function on $[a,b]$ which is differentiable at every point of $[a,b]$. Show that the derivative f' is also measurable.

3. Let X be a locally compact Hausdorff space, $C_c(X)$ the vector space of continuous functions $X \to \mathbf{R}$ having compact support. Suppose $\mu_1, \mu_2, ...$ are Borel measures (each finite on compact sets) on X, and suppose that for each $f \in C_c(X)$ the limit $\lim_{n\to\infty} \int f\, d\mu_n$ exists (as a finite real number). Explain why there is a Borel measure μ on X such that

$$\lim_{n\to\infty} \int_X f\, d\mu_n = \int_X f\, d\mu$$

   for every $f \in C_c(X)$.

4. (i) State the Monotone Convergence Theorem.

   (ii) Determine $\lim_{t\downarrow 0} \int_X e^{-|f(x)|^2/t}\, d\mu(x)$ where f is a measurable function on a measure space $(X, \mathcal{B}, \mu)$ for which $\mu(X) < \infty$.

   (iii) Give an example of a sequence of bounded, non-negative, measurable functions f_n on some measure space $(X, \mathcal{B}, \mu)$, with $f_1 \geq f_2 \geq \cdots$ such that $\lim_{n\to\infty} \int f_n\, d\mu > \int \lim_{n\to\infty} f_n\, d\mu$.

5. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let f be a measurable function on X such that $|f|_\infty \neq 0$ (i.e. f isn't 0 a.e.).
   (i) Suppose $|f|_p$ and $|f|_q < \infty$ for some $p, q \in (0, \infty)$. Show that $f \in L^r(\mu)$ for every number r between p and q.
   (ii) Show that the function $x \mapsto x \log |f|_x$ is convex on any open interval of values of $x \in (0, \infty)$ for which $|f|_x$ is finite.

6. Let $f, g \in L^1(\mathbf{R})$, where we use Lebesgue measure on $\mathbf{R}$.
   (i) Show that

$$\int_{\mathbf{R}} \left[\int_{\mathbf{R}} |f(x-y)g(y)|\, dy \right] dx = |f|_1 |g|_1$$

   where $|\cdot|_1$ denotes the L^1–norm.
   (ii) Explain why the integral

$$h(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}} f(x-y)g(y)\, dy$$

   exists for almost every $x \in \mathbf{R}$ and specifies h as an almost-everywhere defined measurable function.
   (iii) Show that $|h|_1 \leq |f|_1 |g|_1$.

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7. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $p, q \in [1, \infty]$ be conjugate indices (i.e. $p^{-1} + q^{-1} = 1$ with the understanding that $\infty^{-1} = 0$). If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, define

$$\Lambda_g(f) = \int_X fg\, d\mu$$

   (i) Explain why the integral defining $\Lambda_g f$ exists.
   (ii) Show that Λ_g is a bounded linear functional on L^p with norm $|\Lambda_g| \leq |g|_q$.
   (iii) Assume $p \in (1, \infty)$, q the conjugate index, and $g \in L^q(\mu)$. Produce a function f for which $\Lambda_g(f) = |f|_p |g|_q$.

8. Suppose λ and μ are finite measures on a sigma-algebra, and suppose $\lambda << \mu$, i.e. λ is absolutely continuous with respect to μ. Show that for any $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\lambda(E) < \epsilon$ for any measurable set E. Is it necessary that both λ and μ be finite measures?