Core 2 Exam: Measure and Integration
August 2003

Instructions: Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours. Good luck!

1. Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map, and let λ denote Lebesgue measure in $\mathbf{R}^n$. Recall that λ is the unique measure defined on the Borel σ–algebra of $\mathbf{R}^n$ which is translation-invariant and satisfies $\lambda([0,1]^n) = 1$. Use this specification of λ to prove that
$$\lambda\big(A(E)\big) = |\det A|\lambda(E)$$
for every Borel $E \subset \mathbf{R}^n$, where $\det A$ is the determinant of the matrix for A.

2. Suppose $(X, \mathcal{B})$ is a measurable space and μ, ν measures on $\mathcal{B}$, and g a non-negative measurable function on X such that
$$\nu(E) = \int_E g\, d\mu$$
for every $E \in \mathcal{B}$. Prove that
$$\int f\, d\nu = \int fg\, d\mu$$
for every non-negative measurable function f on X.

3. Let H be a Hilbert space.
   (i) Suppose $x_1, x_2, ... \in H$ are such that each x_n lies in the closed unit ball in H and, moreover, the mid-points $(x_n + x_m)/2$ approach 1 in norm, as n, m go to infinity, i.e. $\lim_{n,m\to\infty} \|(x_n + x_m)/2\| = 1$. Prove that the limit $\lim_{n\to\infty} x_n$ exists.

   (ii) Let K be a non-empty closed convex subset of H and $p \in H$. Prove that there is a unique point in K which is closest to p.

4. State very short answers/explanations for the following:
   (i) The Lebesgue measure of the set $E = \{(x, y) \in \mathbf{R}^2 : x^2 - y^3 = 1\}$ is 0. Explain why.
   (ii) Suppose f is a measurable function on $\mathbf{R}$ such that $\int_{\mathbf{R}} |f(t)|\, dt < \infty$. Explain why
$$\lim_{T\to\infty} \int_{-T}^{T} f(t)\, dt = \int_{\mathbf{R}} f(t)\, dt$$
   (iii) Let $p, q \in (1, \infty)$ be conjugate indices, and $(X, \mathcal{F}, \mu)$ a measure space. Use the identity $\sup_{g\in L^q, \|g\|_q \le 1} \left| \int fg\, d\mu \right| = \|f\|_p$, to give a short proof of the triangle inequality for the L^p–norm.
   (iv) Let M be a closed subspace of a Hilbert space H, x a point in H and x' the point on M closest to x. What can you say about the vector $x - x'$ in relation to the subspace M?

5. Let λ denote Lebesgue measure in $\mathbf{R}^n$, and let K be a compact subset of $\mathbf{R}^n$. Let $f(x) = \lambda\big(K \cap (K+x)\big)$ for $x \in \mathbf{R}^n$.
   (i) Show that for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for all $x \in \mathbf{R}^n$ of length $\|x\|$ less than δ. (Hint: Approximate K in measure within ϵ from the outside by an open set U. Choose δ in such a way that $K + x$ is contained in U for $\|x\| < \delta$. Consider the measure of $K \cup (K + x)$.)
   (ii) Use the result in (i) to show that if $\lambda(K) > 0$ then there is a $\delta > 0$ such that for any vector x of length less than δ, the set K contains two points a and b with $a - b = x$.

6. Let X be a locally compact Hausdorff space, $C_c(X)$ the vector space of continuous functions $X \to \mathbf{R}$ having compact support. Suppose μ is a Radon measure on X, i.e. a measure on the Borel σ–algebra of X which is finite on compact sets and satisfies standard regularity properties. Prove that

$$\inf_{1_K \leq f \in C_c(X)} \int_X f \, d\mu = \mu(K)$$

where the infimum is over all continuous functions f of compact support satisfying $f \geq 1_K$.

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7. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $p, q \in (1, \infty)$ be conjugate indices, i.e. p, q satisfy $p^{-1} + q^{-1} = 1$.
   (i) Prove Hölder's inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

   for all $f \in L^p(\mu)$ and $g \in L^q(\mu)$.
   (ii) Assuming that $f \in L^p(\mu)$ and $1 < p < \infty$, construct a $g \in L^q(\mu)$ with $\|g\|_q \leq 1$ for which $\int fg \, d\mu = \|f\|_p$.

8. Let V be a real vector space and p a convex functional on V, i.e. p is a mapping $V \to [0, \infty)$ such that $p(tv) = |t| p(v)$ and $p(v + w) \leq p(v) + p(w)$ for all $v, w \in V$ and $t \in \mathbf{R}$. Let W be a subspace of V, and $g : W \to \mathbf{R}$ a linear functional such that $g(w) \leq p(w)$ for all $w \in W$. If y is a vector in V outside W show that there is a linear functional $f : W + \mathbf{R}y \to \mathbf{R}$ such that $f|W = g$ and $f \leq p$ on $W + \mathbf{R}y$.