Instructions: *Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours.* Good luck!

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1. Let $(X, \mathcal{B}, \mu)$ be a measure space, and $A_1, A_2, ... \in \mathcal{B}$. Prove that
$$\mu\left(\cup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \mu(A_n)$$

2. Suppose $f_1, f_2, ... \in L^p(\mu)$, is a sequence of functions on some measure space $(X, \mathcal{F}, \mu)$, where $1 \leq p < \infty$, such that there is some $g \in L^p(\mu)$ with $|f_n(x)| \leq g(x)$ for almost every $x \in X$, and suppose
$$f(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n(x)$$
exists for almost every $x \in X$. Prove that $f \in L^p(\mu)$ and
$$\|f_n - f\|_p \to 0 \qquad \text{as } n \to \infty$$

3. Show that
$$\int_{\mathbf{R}} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$
where $a \in \mathbf{C}$ and $\mathrm{Re}(a) > 0$, and the square-root on the right is the one with positive real part. Justify your steps by pointing out at each relevant step which basic theorem (such as dominated convergence or Fubini, for example) you are using.

4. If x, y are elements in a complex Hilbert space prove that the inner-product can be computed from the norm function by:
$$\langle x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} \|x + e^{i\theta}y\|^2 e^{i\theta}\, d\theta$$

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5. Prove that the set $C_c^\infty(\mathbf{R}^N)$ of all C^∞ functions of compact support on $\mathbf{R}^N$ is a dense subspace of $L^p(\mathbf{R}^N)$ for each $p \in [1, \infty)$.

6. 6. Prove that if μ is a Radon measure on a locally compact Hausdorff space X then $C_c(X)$ is a dense subspace of $L^p(X, \mu)$, for every $p \in [1, \infty)$.

7. If λ, μ, and ν are measures on a measurable space $(X, \mathcal{B})$ and if

$$\lambda << \mu << \nu$$

prove the chain rule:

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}\frac{d\mu}{d\nu}$$

8. Prove the change of variables formula

$$\int_0^\infty f(x)\,dx = \int_0^\infty f\big(\phi(x)\big)\phi'(x)\,dx$$

for $f \in L^1(\mathbf{R})$, where $\phi : [0, \infty) \to [0, \infty)$ is differentiable with continuous positive derivative $\phi' > 0$.