Instructions: *Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours.* Good luck!

---

1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to [0, \infty]$ a measurable function.
   (i) Show that if $\int_E f \, d\mu = 0$ for all $E \in \mathcal{B}$ then $f = 0$ μ-almost-everywhere.
   (ii) Show that if $\int_E f \, d\mu < \infty$ for all $E \in \mathcal{B}$ then $f < \infty$ μ-almost-everywhere.

2. Let $(X, \mathcal{F})$ be a measurable space, and let H be the set of all measurable functions $X \to \mathbf{R}$. Show that
$$\sigma(H) = \mathcal{F},$$
where $\sigma(H)$ is the smallest σ-algebra with respect to which all functions in H are measurable.

3. Let f be a measurable function on a measure space $(X, \mathcal{F}, \mu)$. Show that if $\|f\|_p < \infty$ and $\|f\|_q < \infty$, where $p, q \geq 1$, then $\|f\|_r < \infty$ for every $r \in (p, q)$.

4. Mark the following statements True or False, and provide a brief explanation for your answer:
   (i) If (f_n) is a sequence of bounded measurable functions on a measure space $(X, \mathcal{B}, \mu)$ and $f_n(x) \to f(x)$ for all $x \in X$, where f is a bounded measurable function, then $\int f_n \, d\mu \to \int f \, d\mu$, as $n \to \infty$.
   (ii) If f_n is a sequence of measurable functions on a measure space converging pointwise to a function f, and $1 \leq p < \infty$, and if $M \geq \|f_n\|_p$ for all $n \in \{1, 2, ...\}$, then $\|f\|_p \leq M$.
   (iii) If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces, and $E \subset X \times Y$ is such that the projections of E on X and on Y are measurable, then E is in the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$.
   (iv) Suppose μ and ν are measures on the same measurable space $(X, \mathcal{B})$, and $\mu << \nu$. If $f, g : X \to \mathbf{R}$ are equal to each other μ-almost-everywhere then they are equal ν-almost-everywhere.
   (v) If H is an infinite dimensional Hilbert space then there is a subspace $M \subset H$ which is dense in H but $M \neq H$.

5. Compute the Lebesgue measure of the ellipsoidal region in $\mathbf{R}^N$ given by

$$E = \left\{ x \in \mathbf{R}^N : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_N^2}{a_N^2} \leq 1 \right\},$$

where $a_1, ..., a_N$ are positive real numbers. Indicate clearly any standard results you use.

6. Show that if $a, b \in \mathbf{C}$, and a has positive real part, then

$$\int_{\mathbf{R}} e^{-ax^2+2bx}\, dx = (\pi/a)^{\frac{1}{2}} \exp(b^2/a),$$

where $(\pi/a)^{1/2}$ is the square root of π/a with positive real part.

---

7. Let X and Y be topological spaces with countable bases. Show that

$$\mathcal{B}_{X\times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y,$$

where $\mathcal{B}_S$ denotes the Borel σ-algebra of S, and the σ-algebra on the right is the product σ-algebra of $\mathcal{B}_X$ with $\mathcal{B}_Y$.

8. Let μ be a Radon measure on a locally compact Hausdorff space X. Prove that

$$\mu(K) = \inf_{1_K \leq f \in C_c(X)} \int_K f\, d\mu.$$