Core 2 Exam: Measure and Integration
August 2004

Instructions: Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours. Good luck!

1. Let \( m \) denote the Lebesgue measure on \( \mathbb{R}^n \), and \( \mathcal{F} \), the \( \sigma \)-algebra of Lebesgue measurable sets in \( \mathbb{R}^n \). Show that for any linear map \( T \) of \( \mathbb{R}^n \) to \( \mathbb{R}^n \), there exists a constant \( C_T \) such that
   \[
   m(T(A)) = C_T m(A) \quad \forall A \in \mathcal{F}
   \]
   where \( T(A) \) is the image of the set \( A \) under \( T \).

2. Let \( X \) be a compact Hausdorff space, and suppose \( \Lambda \) is a positive linear functional on \( C(X) \). Show that if \( f_n \to f \) uniformly on \( X \), then \( \Lambda f_n \to \Lambda f \). That is, a positive linear functional on \( C(X) \) is continuous in the norm topology:
   \[
   ||f||_\infty = \sup\{|f(x)| : x \in X\}.
   \]
   Give an example to show that the above result doesn’t hold in general if we only require \( X \) to be locally compact.

3. If \( \mu \) and \( \nu \) are finite measures on a \( \sigma \)-algebra \( \mathcal{F} \), show that \( \nu \) is absolutely continuous with respect to \( \mu \) if and only if for any given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \nu(A) < \epsilon \) for all \( A \) with \( \mu(A) < \delta \).

4. Let \((X, \mathcal{F}, \mu)\) be a \( \sigma \)-finite measure space such that for any \( n \geq 1 \), \( f_n \in L^p(\mu) \) where \( p \geq 1 \). Suppose \( ||f_n - f||_p \to 0 \) and \( f_n \to g \) a.e. as \( n \to \infty \). Is there a relationship between \( f \) and \( g \)? Prove your claim(s), or give counter-example(s).

5. If \( f \) is a positive Lebesgue measurable function on \([0,1]\), which is larger:
   \[
   \int_0^1 f(x) \log f(x) m(dx) \quad \text{or} \quad \int_0^1 f(x) m(dx) \int_0^1 \log f(x) m(dx).
   \]
   Prove your assertion.
6. Show that if $h$ is a bounded continuous function mapping $[0,1] \to \mathbb{R}$, show that
\[
\lim_{n \to \infty} \frac{1}{n} \log \int_0^1 e^{-nh(x)} \, dx = - \min_{x \in [0,1]} h(x).
\]

7. For $x \in \mathbb{R}$ and $t > 0$, let
\[
f(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.
\]
It is well-known that for each $t > 0$, $\int_{-\infty}^{\infty} f(x, t) \, dx = 1$. It is also known that
\[
2 \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.
\]
If $g(x, t) = \frac{\partial f}{\partial t}$, prove or disprove:
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, t) \, dx \, dt \neq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, t) \, dx \, dt.
\]
What is the relevance of this example to the Fubini theorem?

8. Let $(a, b)$ be a finite non-empty interval. Show that an orthonormal sequence $\{e_n\}$ is a basis for the real Hilbert space $L^2(a, b)$ if the following holds:
\[
\sum_{n=1}^{\infty} (\int_a^x e_n(t) \, m(dt))^2 = x - a
\]
for all $x \in (a, b)$.