Core 2 Exam: Measure and Integration
August 2004

Instructions: Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours. Good luck!

1. Let m denote the Lebesgue measure on $\mathbf{R}^n$, and $\mathcal{F}$, the σ-algebra of Lebesgue measurable sets in $\mathbf{R}^n$. Show that for any linear map T of $\mathbf{R}^n$ to $\mathbf{R}^n$, there exists a constant C_T such that

$$m(T(A)) = C_T\, m(A) \quad \forall A \in \mathcal{F}$$

   where $T(A)$ is the image of the set A under T.

2. Let X be a compact Hausdorff space, and suppose Λ is a positive linear functional on $C(X)$. Show that if $f_n \to f$ uniformly on X, then $\Lambda f_n \to \Lambda f$. That is, a positive linear functional on $C(X)$ is continuous in the norm topology: $||f||_\infty = \sup\{|f(x)| : x \in X\}$.

   Give an example to show that the above result doesn't hold in general if we only require X to be locally compact.

3. If μ and ν are finite measures on a σ-algebra $\mathcal{F}$, show that ν is absolutely continuous with respect to μ if and only if for any given $\epsilon > 0$, there exists a $\delta > 0$ such that $\nu(A) < \epsilon$ for all A with $\mu(A) < \delta$.

4. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space such that for any $n \geq 1$, $f_n \in L^p(\mu)$ where $p \geq 1$. Suppose $||f_n - f||_p \to 0$ and $f_n \to g$ a.e. as $n \to \infty$. Is there a relationship between f and g? Prove your claim(s), or give counter-example(s).

---

5. If f is a positive Lebesgue measurable function on $[0, 1]$, which is larger:

$$\int_0^1 f(x) \log f(x) m(dx) \quad \text{or} \quad \int_0^1 f(x) m(dx) \int_0^1 \log f(x) m(dx).$$

   Prove your assertion.

6. Show that if h is a bounded continuous function mapping $[0,1] \to \mathbf{R}^1$, show that

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 e^{-nh(x)} dx = -\min_{x\in[0,1]} h(x).$$

---

7. For $x \in \mathbf{R}^1$ and $t > 0$, let

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

It is well-known that for each $t > 0$, $\int_{-\infty}^{\infty} f(x,t)dx = 1$. It is also known that $2\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$.

If $g(x,t) = \frac{\partial f}{\partial t}$, prove or disprove:

$$\int_{-\infty}^{\infty} \int_s^{\infty} g(x,t)dt\, dx \neq \int_s^{\infty} \int_{-\infty}^{\infty} g(x,t)dx\, dt.$$

What is the relevance of this example to the Fubini theorem?

8. Let (a,b) be a finite non-empty interval. Show that an orthonormal sequence $\{e_n\}$ is a basis for the real Hilbert space $L^2(a,b)$ if the following holds:

$$\sum_{n=1}^{\infty} (\int_a^x e_n(t)m(dt))^2 = x - a$$

for all $x \in (a,b)$.