Core 2 Exam: Measure and Integration
January 2003

Instructions: Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). You have three and a half hours. Good luck!

1. Calculate the volume (Lebesgue measure) of the subset D_n of $\mathbf{R}^n$ consisting of all points $x = (x_1, ...x_n) \in \mathbf{R}^n$ for which $\sum_{j=1}^n x_j \leq 1$ and $x_1, ..., x_n \geq 0$.

2. In this problem you may use the Gaussian integral formula

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}} \, dx = 1.$$

(i) Show that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}+tx} \, dx = e^{t^2/2} \tag{1}$$

for all $t \in \mathbf{R}$.

(ii) Prove that

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2} x^{2n} \, dx = e^{t^2/2} \tag{2}$$

holds for $t \in \mathbf{R}$. [You may use equation (1).]

3. Let H be a complex Hilbert space.

(i) Show that the inner-product on H can be expresses using the norm in the following way:

$$(x, y) = \int_0^1 |x + e^{2\pi i t} y|^2 e^{2\pi i t} \, dt \qquad \text{for all } x, y \in H.$$

(ii) If $x_1, x_2, ...$ are orthogonal vectors in H shows that the series $\sum_{n=1}^{\infty} x_n$ converges in H if and only $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. (Hint: Let $s_n = x_1 + \cdots + x_n$, and work out $|s_n - s_m|^2$.)

4. (i) State the Dominated Convergence Theorem.

(ii) Prove that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}+zx} \, dx = e^{z^2/2}$$

holds for all $z \in \mathbf{C}$. [You may use the results from Problem 2.]

5. Let X be a locally compact Hausdorff space and μ a measure on the Borel sigma-algebra such that: (i) $\mu(K) < \infty$ for all compact $K \subset X$, (ii) every open set is μ–inner-regular and every Borel set is μ–outer-regular, i.e.

$$\mu(U) = \sup\{\mu(K) : \text{compact } K \subset U\} \qquad \text{for all open } U \subset X \text{ and}$$

$$\mu(E) = \inf\{\mu(U) : \text{open } U \supset E\} \qquad \text{for all Borel } E \subset X$$

Suppose ν is a measure on the Borel σ–algebra of X satisfying the same conditions (i) and (ii), i.e. ν is finite on compact sets, every open set is ν–inner-regular and every Borel set is ν–outer-regular. Assume, furthermore, that $\int f\, d\mu = \int f\, d\nu$ for every continuous function on X having compact support. Prove that $\mu = \nu$.

6. Let X and Y be normed linear spaces and $f : X \to Y$ a linear mapping. Show that f is continuous if and only if f maps the unit ball of X into a bounded subset of Y (i.e. $\sup_{x \in X, |x| \leq 1} |f(x)| < \infty$).

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7. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $p, q \in (1, \infty)$ be conjugate indices (i.e. $p^{-1} + q^{-1} = 1$). For $f \in L^p(\mu)$ and $g \in L^q(\mu)$, define

$$\Lambda_f(g) = \int_X fg\, d\mu$$

The Hölder inequality implies that the integral $\int_X fg\, d\mu$ exists and $\Lambda_f : L^q(\mu) \to \mathbf{C}$ is a bounded linear functional (you do not have to prove this). Let $f \in L^p$ and let g be the function on X given by

$$g(x) = \begin{cases} \frac{|f(x)|^p}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

(i) Work out the L^q–norm $|g|_q$ of g.
(ii) Show that $\Lambda_f g$ equals $|f|_p^p$.
(iii) Prove that the norm $|\Lambda_f|$ equals $|f|_p$.

8. Suppose μ is a finite measure on a σ-algebra $\mathcal{B}$ of subsets of a set X, and $f : X \to [0, \infty)$ is a measurable function for which $\int f\, d\mu < \infty$. Show that for any $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E f\, d\mu < \epsilon$ for any measurable set E.