Core 2 Exam: Measure and Integration
January 2005

Instructions: *Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems).* The problems carry equal weight. You have three and a half hours. Good luck!

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_k\}$ be a a sequence of measurable sets in X satisfying $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Prove that μ - almost all $x \in X$ belong to at most finitely many of the sets E_k.

2(a) Write the full statement of the Riesz representation theorem for positive linear functionals defined on $C_c(X)$ where X is a locally compact Hausdorff space.

(b) Let Λ be a positive linear functional on $C_c(X)$, and let there exist two measures (satisfying the properties of measures listed in the Riesz representation theorem) μ_1 and μ_2 such that $\Lambda(f) = \int_X f\, d\mu_1 = \int_X f\, d\mu_2$ for all $f \in C_c(X)$. Show that $\mu_1(K) = \mu_2(K)$ for all K compact in X.

Hint: Apply the Urysohn Lemma.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Show that $||f||_{L^p}$ increases to $||f||_\infty$ as p tends to ∞.

4. Let μ be a measure on a measure space $(X, \mathcal{M})$, and let ν be a signed measure on $(X, \mathcal{M})$.

(i) Show that if $\nu << \mu$, then $|\nu| << \mu$.

(ii) If ν is such that $\nu \perp \mu$, and $\nu << \mu$, show that $\nu = 0$.

---

5. Let $\mathbf{N}$ denote the set of all positive integers (not including 0). Let $X = Y = \mathbf{N}$. Let $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbf{N})$ where $\mathcal{P}$ denotes the power set. Let $\mu = \nu =$ counting measure on $\mathbf{N}$. Consider the product measure space $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$. Define the function f on $X \times Y$ by

$$f(m,n) = \begin{cases} 1, & \text{if } m = n; \\ -1, & \text{if } m = n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Find each of the following:

(i) $\int_Y \int_X f d\mu \, d\nu$

(ii) $\int_X \int_Y f \, d\nu \, d\mu$

(iii) $\int_{X \times Y} |f| \, d(\mu \times \nu)$ Does this example satisfy the conditions of the Fubini theorem? Explain.

6. Suppose μ and ν are finite measures on $(X, \mathcal{M})$ with $\nu << \mu$, and define $\lambda = \mu + \nu$. If $f = \dfrac{d\nu}{d\lambda}$, then show that

$$0 \leq f < 1 \ \mu - \text{a.e. and } \frac{d\nu}{d\mu} = \frac{f}{(1-f)}$$

---

7. Let K be a closed convex set in a Hilbert space H. Show that there is a unique element x_0 in K such that $||x_0|| \leq ||x||$ for all $x \in K$.

8. The distribution function of a non-negative measurable function f on a measurable space $(X, \mathcal{M})$ with respect to a measure μ is defined by $\lambda_f(t) = \mu(\{x : f(x) > t\})$. Prove that

(i) $\int_X f \, d\mu = \int_0^\infty \lambda_f(t) dt$ and in general,

(ii) for any finite $p \geq 1$, $\int_X f^p \, d\mu = \int_0^\infty pt^{p-1} \lambda_f(t) dt$.