Instructions: *Do Problems 1-4, one of Problems 5 and 6, and one of Problems 7 and 8 (a total of six problems). Write your name clearly at the top of each sheet of paper you turn in and write the number of the problem being solved. You have three and a half hours.* Good luck!

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1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to [0, \infty]$ a measurable function. Define
$$\nu(A) = \int_A f \, d\mu$$
for every $A \in \mathcal{B}$. Prove that ν is a measure.

2. Let $f \in L^1(\mathbf{R})$. Prove that
$$\lim_{T\to\infty} \int_T^\infty |f(x)| \, dx = 0.$$

3. Let $f, g \in L^1(\mathbf{R}^N)$, with respect to Lebesgue measure on $\mathbf{R}^N$. Prove that
$$\int_{\mathbf{R}^N} |f(x-y)g(y)| \, dy < \infty \qquad \text{for almost every } x \in \mathbf{R}^N.$$
Explain your reasoning, indicating clearly any standard results you might use.

4. Let H be a Hilbert space, with inner-product denoted by $\langle \cdot, \cdot \rangle$. Prove that for any $y \in H$, the mapping
$$H \to \mathbf{C} : x \mapsto \langle x, y \rangle$$
is continuous.

5. Compute the volume (i.e. Lebesgue measure) of the unit ball $\{x \in \mathbf{R}^N : \|x\| \leq 1\}$ in $\mathbf{R}^N$, indicating clearly any standard results you might use.

6. Let $(X, \mathcal{B}, \nu)$ be a measure space with $\nu(X) < \infty$. Suppose g is a measurable function on $(X, \mathcal{B})$ with the property that for every $h \in L^1(\nu)$ the function gh is integrable and
$$\left| \int gh \, d\nu \right| \leq \|h\|_1.$$
Prove that $\|g\|_\infty \leq 1$. [Hint: Consider the set $E_t = \{x \in X : |g(x)| > t\}$, where t is any real number > 1; choose a function h in terms of E_t and show that $\nu(E_t)$ would have to be 0.]

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7. Let $(X, \mathcal{B}, \mu)$ be a measure space and $f : X \to [0, \infty]$ a measurable function. Define
$$\nu(A) = \int_A f \, d\mu$$
for every $A \in \mathcal{B}$. Now take for granted that ν is a measure. Prove that
$$\int \phi \, d\nu = \int f\phi \, d\mu$$
for every non-negative measurable function ϕ on $(X, \mathcal{B})$.

8. Suppose μ and ν are Radon measures on a locally compact Hausdorff space X such that
$$\int f \, d\mu = \int f \, d\nu$$
for every continuous function f on X having compact support. Prove that $\mu = \nu$.