August, 2007

# Comprehensive Exam for Ordinary Differential Equations

INSTRUCTIONS: Choose and do 100 points; you can do 20 more points for extra credit. You have 3 1/2 hours to complete the exam. ¡Buena suerte!

1. (10 points) Let $f : R \to \mathbb{R}$, where $R$ is the square $|x| \leq a$, $|y| \leq a$, satisfy the conditions
$$f(x,y) < 0 \text{ if } xy > 0 , \ f(x,y) > 0 \text{ if } xy < 0 .$$
Find all solutions of the initial value problem $y' = f(x,y)$, $y(0) = 0$.

2. (20 points) Let $f(t)$ be continuous in $[1,\infty)$ be such that $\int_1^\infty t\,|f(t)|\,dt < \infty$. Prove that the equation
$$x''(t) + f(t)\,x(t) = 0 ,$$
has a solution $x_1$ such that
$$\lim_{t\to\infty} x_1(t) = 1 , \lim_{t\to\infty} x_1'(t) = 0 .$$
[Hint: use succesive approximations in the equation
$$x_1(t) = 1 + \int_t^\infty (t-s)\,f(s)\,x_1(s)\,ds .]$$

3. (10 points) Solve
$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} .$$

4. (10 points) Solve
$$(x+z)\frac{\partial z}{\partial x} + (y+z)\frac{\partial z}{\partial y} + z = 0 .$$

5. (10 points) Prove that if $\mu_1(x,y)$ and $\mu_2(x,y)$ are two integration factors of the equation $M(x,y)\,dx + N(x,y)\,dy = 0$, and if $F = \mu_1/\mu_2$ is not constant then $F$ is a first integral.

6. (10 points) Find a function $f(x,y)$ such that $f(0,y) = (1+y^2)^{-1}$ and such that

$$\int_C f(x,y) \left(2xy\,dx + \left(1-x^2\right)\,dy\right) = 0\ ,$$

for each closed curve $C$.

7. (15 points) Solve the matrix differential equation $x' = Ax$, the vector equation $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} e^t \\ 2e^{2t} \end{pmatrix}$$

8. (10 points each) Solve the equation for $y(x)$ :

**a)** $y'' - 4y' + 4y = xe^x$,

**b)** $x^2y'' + xy' - \alpha^2 y = x^\beta$, where $\alpha$ and $\beta$ are constants.

9. (20 points)

(a) Let $y$ be a non trivial solution of a linear homogeneous equation of order $n$ in the interval $(a,b)$. If $[c,d] \subset (a,b)$, show that $y$ has a finite number of zeros in $[c,d]$. Give an example of an equation and a solution $y$ that has infinite zeros in $(a,b)$.

(b) Let $y_1(t)$ and $y_2(t)$ be a basis of the solution space of the equation

$$y''(t) + a(t)\,y'(t) + b(t)\,y(t) = 0\,.$$

Prove that the zeros of $y_1$ and of $y_2$ alternate: between two zeros of $y_1$ there is exactly one zero of $y_2$.

(c) Let $y$ be a solution of $y'' + b(t)\,y = 0$. If $b(t) < 0$, then $y(t)$ has at most one zero.

10. (20 points)

    (a) Define $\cos A$ and $\sin A$ if $A$ is an $n \times n$ matrix, i.e., $A \in \mathcal{M}_{n\times n}$.

    (b) Compute $(\cos A(t))'$ and $(\sin A(t))'$ if $A(t)$ is differentiable and commutes with $A'(t)$.

    (c) Find $\cos\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $\sin\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

    (d) Prove that all solutions of the equation $x'' + A^2 x = 0$ are of the form $x = (\cos At)\, c_1 + (\sin At)\, c_2$ with $c_1$ and $c_2$ arbitrary $n \times n$ matrices. Solve the equation in case $A = \begin{pmatrix} 7\pi & 12\pi \\ -4\pi & -7\pi \end{pmatrix}$.

11. (15 points)

    (a) Let $y_1, \ldots, y_n \in C^n(a,b)$. Show that there exists a normal homogeneous differential equation of order $n$ for which they are a basis of the solution space if and only if their Wronskian satisfies $W[y_1(t), \ldots, y_n(t)] \neq 0\ \forall t \in (a,b)$.

    (b) Show that the equation is unique if the coefficient of $y^{(n)}$ is 1.

    (c) Find the equation in case $y_1(t) = \sec^2 t$ and $y_2(t) = \tan^2 t$.

12. (15 points)

    (a) Solve the system

$$z\omega_1'(z) = -\omega_2(z)\ ,$$
$$z\omega_2'(z) = \omega_1(z)\ .$$

    (b) Find $z^A$ if $A = \begin{pmatrix} 8 & 8 \\ -2 & 0 \end{pmatrix}$.

13. (15 points) The equation $z^2 y''(z) + zp(z)\, y'(z) + q(z)\, y(z) = 0$ becomes the system $z\mathbf{w}'(z) = A(z)\,\mathbf{w}(z)$ under the change $\mathbf{w} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, $\omega_1 = y$, $\omega_2 = zy'$.

    (a) Find $A$ explicitly if you start with Bessel's equation.

    (b) Start with the equation $z^2 y'' - zy' + 2y = 0$, transform to a system, and solve the system.

14. (10 points) Find all Frobenious series solutions of the equation $z^4 y'' + y = 0$.