August, 2004

Comprehensive Exam
for
Ordinary Differential Equations

INSTRUCTIONS: Choose and do 100 points; you can do 20 more points for extra credit. You have 31/2 hours to complete the exam. ¡Buena suerte!

1. (15 points) Let f be continuous in $[1,\infty)$ with $\int_1^\infty t\,|f(t)|\,\mathrm{d}t < \infty$. Prove that the equation

$$y''(t) + f(t)\,y(t) = 0\,,$$

has a unique solution that satisfies

$$\lim_{t\to\infty} y(t) = 1\,, \qquad \lim_{t\to\infty} y'(t) = 0\,.$$

[Hint: use successive approximations in $y(t) = 1 + \int_t^\infty (t-s)\,f(s)\,y(s)\,\mathrm{d}s$.]

2. (20 points) **a)** Define $\cos A$ and $\sin A$ if A is an $n\times n$ matrix, i.e., $A \in \mathcal{M}_{n\times n}$.
b) Compute $(\cos A(t))'$ and $(\sin A(t))'$ if $A(t)$ is differentiable and commutes with $A'(t)$.
c) Find $\cos\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $\sin\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.
d) Prove that all solutions of the equation $x'' + A^2x = 0$ are of the form $x = (\cos At)\,c_1 + (\sin At)\,c_2$ with c_1 and c_2 arbitrary $n\times n$ matrices. Solve the equation in case $A = \begin{pmatrix} 7\pi & 12\pi \\ -4\pi & -7\pi \end{pmatrix}$.

3. (15 points) **a)** Let $y_1,\ldots,y_n \in C^n(a,b)$. Show that there exists a normal homogeneous differential equation of order n for which they are a basis of the solution space if and only if their Wronskian satisfies $W[y_1(t),\ldots,y_n(t)] \neq 0\ \forall t\in(a,b)$.
b) Show that the equation is unique if the coefficient of $y^{(n)}$ is 1.
c) Find the equation in case $y_1(t) = \sec^2 t$ and $y_2(t) = \tan^2 t$.

4. (20 points) **a)** Let $y(t)$ be a solution of the ordinary differential equation

$$y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_0(t)\,y(t) = 0\,,$$

in the interval (a,b). Suppose that $[c,d] \subset (a,b)$. Prove that if $y \neq 0$ then it has a finite number of zeros in $[c,d]$. Is this true in (a,b)?

b) Let $y_1(t)$ and $y_2(t)$ be a basis of the solution space of the equation

$$y''(t) + a(t)\,y'(t) + b(t)\,y(t) = 0\,.$$

Prove that the zeros of y_1 and of y_2 alternate: between two zeros of y_1 there is exactly one zero of y_2.

c) Let y be a solution of $y'' + b(t)\,y = 0$. If $b(t) < 0$, then $y(t)$ has at most one zero.

5. (10 points) Let $u_1(x,y)$ and $u_2(x,y)$ be two integrating factors of the equation

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0\,. \tag{1}$$

Suppose $F(x,y) = u_1(x,y)/u_2(x,y)$ does not reduce to a constant. Show that the solution of (**??**) is given by $F(x,y) = c$, c constant.

6. (10 points) Let $T : C[0,2\pi] \longrightarrow C[0,2\pi]$ be given by $T(f)(x) = \int_0^x f(s)\cos s\,\mathrm{d}s$. Consider the norm $\|\ \|_\infty$ in $C[0,2\pi]$. Find the operator norm of T.

7. (15 points) **a)** Solve the system

$$\begin{aligned} z\omega_1'(z) &= -\omega_2(z)\,, \\ z\omega_2'(z) &= \omega_1(z)\,. \end{aligned}$$

b) Find z^A if $A = \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix}$.

8. (15 points) The equation $z^2y''(z) + zp(z)\,y'(z) + q(z)\,y(z) = 0$ becomes the system $z\mathbf{w}'(z) = A(z)\,\mathbf{w}(z)$ under the change $\mathbf{w} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$, $\omega_1 = y$, $\omega_2 = zy'$.

a) Find A explicitly if you start with Bessel's equation.

b) Start with the equation $z^2y'' - zy' + 2y = 0$, transform to a system, and solve the system.

9. (20 points) **a)** Find all Frobenious series solutions of the equation $z^4y'' + y = 0$.

b) Give an example of an equation with an irregular singular point at $z = 0$ that has one non-trivial Frobenious series solution.

c) If a homogeneous second order equation has two linearly independent Frobenious series solutions is it true that the point is a regular singular point?

10. (10 points) Show that Laguerre's equation, $zy'' + (1-z)\,y' + \lambda y = 0$ has an entire solution for each $\lambda \in \mathbb{C}$. Show that this solution becomes a polynomial if $\lambda \in \mathbb{N}$.