Math 7512 Topology II - Core II Exam August, 2004

Do ***any five*** *of the following problems. Indicate clearly which five problems you have selected. You have three and one-half hours to complete the test. Good luck!*

1. State the Seifert-van Kampen theorem and use it to compute the fundamental group of the projective plane, $\mathbb{R}P^2$.

2. Let X be Euclidean 3-space with the following two lines deleted, $X = \mathbb{R}^3 \backslash \{(t, 0, 1), (t, 0, -1) | t \in \mathbb{R}\}$. Compute the fundamental group of X. (Be sure to justify your computation).

3. Let $X = S^1 \vee S^1$, the wedge of two circles joined at a single point. Define *universal covering* and either explicitly describe a universal cover of X or justify why it must exist. Exhibit both a regular and an irregular 3-fold covering of X.

4. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

5. Give an example of a space whose fundamental group is a cyclic group of order six. Provide a proof that the fundamental group is cyclic.

6. Give an explicit construction of the free group on two generators. State its universal property. Construct a finite state automata that provides a normal form for this group. Define the growth function and describe a method for its computation (you do not need to carry out the method) .

7. For each pair of spaces (X, A) in the following list, determine whether a retraction $r : X \mapsto A$ exists or not. If the retraction exists, sketch a construction of such a retraction; if it does not, provide an explanation.

   (a) $X = \mathbb{R}$ and $A = [0, 1]$, a closed interval.

   (b) $X = \mathbb{R}^2 \backslash (0, 0)$, the punctured plane and $A = (0, 1)$, a single point.

   (c) $X = D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}, A = S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

   (d) X is the Möbius band and A is the boundary circle.