Math 7512 Topology II - Core II Exam January, 2005

Do ***any five*** *of the following problems. Indicate clearly which five problems you have selected.* You have three and one-half hours to complete the test. Good Luck!

1. Compute the fundamental group of the open subset X of $\mathbb{R}^3$ obtained by removing the three coordinate axes.

2. Prove that there is no open cover, $\{U, V\}$, of the real projective plane, $\mathbb{R}P^2$, with two connected open sets such that U and V are both contractible and $U \cap V$ is connected.

3. State precisely the Seifert-van Kampen theorem and use it to compute the fundamental group of the connected sum of two projective planes, $\mathbb{R}P^2$ (equivalently, the space obtained by identifying two Möbius bands along the boundary circle). Describe all the regular covering spaces.

4. Prove the Brouwer fixed point theorem in dimension two: Every continuous map $f : D^2 \to D^2$ has a fixed point.

5. Show that the following three conditions (on a topological space X) are equivalent:

   (a) Every map $S^1 \to X$ is homotopic to a constant map.

   (b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

   (c) The fundamental group, $\pi_1(X, x_0)$, is trivial for all $x_0 \in X$.

6. Let $\{f_i, g_i : i = 0, 1\}$ be four closed paths based at x_0. Define the concept of path-homotopy (denoted $\simeq$) and also the operation of path multiplication (denoted $f_0 \circ f_1$). Show that the following cancelation property holds for path-homotopy , if $f_0 \circ g_0 \simeq f_1 \circ g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

7. Let $p : \widetilde{X} \to X$ be a covering space with path-connected cover. Explicitly define the right action of $\pi_1(X, x)$ on the fiber $p^{-1}(x)$ for a given point $x \in X$. Show that this is a group action which satisfies the transitive property.