Comprehensive Examination: Topology - I

August, 2000

Work any three of the following four problems.

Problem 1. (i) Let $\mathcal{D}$ be a descending family of nonempty compact sets ($D_1, D_2 \in \mathcal{D} \Rightarrow \exists D_3 \in \mathcal{D}$) in a Hausdorff space X, let $F := \bigcap\{D : D \in \mathcal{D}\}$ and let U be an open set containing F. Show that $F \neq \emptyset$ and there exists $D \in \mathcal{D}$ such that $D \subseteq U$.

(ii) Let A_n be a decreasing sequence of closed sets in a complete metric space X such that $\lim_n \operatorname{diam}(A_n) = 0$, where $\operatorname{diam} A_n := \sup\{d(x,y) : x, y \in A_n\}$. Show that $\bigcap_n A_n$ consists of a singleton subset.

Problem 2. (i) Let $f : X \to Y$ be a function from a topological space X to a space Y. We say that f is continuous at $x \in X$ if for every open set V containing $f(x)$, there exists U open containing x such that $f(U) \subseteq V$, and we say that f is continuous if it is continuous at every $x \in X$. Show that f is continuous (in the preceding sense) if and only if for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

(ii) Let $f : X \to Y$ be a continuous function from a compact metric space X into a metric space Y. Show that f is uniformly continuous (given $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_1, x_2) \leq \delta$ implies that $d(f(x_1), f(x_2)) < \epsilon$). You may use freely standard theorems about compactness and about continuous functions.

(iii) Prove or disprove: The function $f(x) = x^2$ from $\mathbb{R}$ into $\mathbb{R}$ is uniformly continuous.

Problem 3. (i) Let X be a Hausdorff space with the property that for every $x \in X$, there exists a compact set K (depending on x) such that $x \in K^\circ$, the interior of K (this is one, although probably not the best, way of defining a locally compact space). Show that the space X is regular.

(ii) Show that a metric space is normal.

Problem 4. Let (X, d) be a metric space.

(i) Show that a sequence $\{x_n\}$ can have a most one point to which it converges (we say that limits are unique).

(ii) A point $y \in X$ is a *cluster point* of $\{x_n\}$ if given $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $m \geq n$ such that $d(x_m, y) < \epsilon$. Show that the set S of cluster points of $\{x_n\}$ is given by

$$S = \bigcap_n \overline{\{x_i |\ i \geq n\}}.$$

(iii) Show that if a Cauchy sequence clusters to a point p, then it converges to p.

(iv) Prove or disprove: If a sequence has exactly one cluster point, then it must converge to that point.