Math 7510 Topology I Core I Exam August 2001

Do **any four** of the following problems. Indicate clearly which four problems you have selected. You have two and a half hours for the test. Good luck!

1. (a) Show that a metric space is normal.
(b) Show that a compact Hausdorff space is normal.

2. Let $\{ X_\alpha \mid \alpha \in J \}$ be a family of topological spaces, and let $X = \prod_{\alpha \in J} X_\alpha$ with the product topology. Let $\pi_\alpha \colon X \to X_\alpha$ be the projection, and let $f \colon Y \to X$ be a function from a topological space Y to X. Prove that f is continuous if and only if the composite $\pi_\alpha \circ f$ is continuous for each $\alpha \in J$.

3. Let X and Y be topological spaces and $f \colon X \to Y$ a quotient mapping onto Y. Let $h \colon X \to Z$ be a continuous mapping to a topological space Z such that $h(x_1) = h(x_2)$ whenever $f(x_1) = f(x_2)$. Show that there is a unique function $g \colon Y \to Z$ such that $g \circ f = h$, and that g is continuous.

4. Let (X, d) be a complete metric space and let $\{A_n\}$ be a nested sequence of non-empty closed subsets of X such that $\lim_{n \to \infty} \operatorname{diam}(A_n) = 0$. Show that there exists $x \in X$ such that $\bigcap_n A_n = \{x\}$.

5. Let C be a connected subset of a topological space X. Prove or disprove:
(a) the closure $\overline{C}$ of C is connected;
(b) the interior C° of C is connected.

6. (a) Show that if $f \colon X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then f is a homeomorphism.
(b) Give an example of topological spaces X and Y and a continuous bijection $f \colon X \to Y$ that is **not** a homeomorphism.