Do any four of the following problems. Indicate clearly which four problems you have selected. You have two and a half hours for the test. Good luck!

1. (a) Show that a metric space is normal.
   (b) Show that a compact Hausdorff space is normal.

2. Let \( \{ X_\alpha \mid \alpha \in J \} \) be a family of topological spaces, and let \( X = \prod_{\alpha \in J} X_\alpha \) with the product topology. Let \( \pi_\alpha: X \to X_\alpha \) be the projection, and let \( f: Y \to X \) be a function from a topological space \( Y \) to \( X \). Prove that \( f \) is continuous if and only if the composite \( \pi_\alpha \circ f \) is continuous for each \( \alpha \in J \).

3. Let \( X \) and \( Y \) be topological spaces and \( f: X \to Y \) a quotient mapping onto \( Y \). Let \( h: X \to Z \) be a continuous mapping to a topological space \( Z \) such that \( h(x_1) = h(x_2) \) whenever \( f(x_1) = f(x_2) \). Show that there is a unique function \( g: Y \to Z \) such that \( g \circ f = h \), and that \( g \) is continuous.

4. Let \((X, d)\) be a complete metric space and let \( \{ A_n \} \) be a nested sequence of non-empty closed subsets of \( X \) such that \( \lim_{n \to \infty} \text{diam}(A_n) = 0 \). Show that there exists \( x \in X \) such that \( \bigcap_n A_n = \{ x \} \).

5. Let \( C \) be a connected subset of a topological space \( X \). Prove or disprove:
   (a) the closure \( \overline{C} \) of \( C \) is connected;
   (b) the interior \( C^\circ \) of \( C \) is connected.

6. (a) Show that if \( f: X \to Y \) is a continuous bijection from a compact space \( X \) to a Hausdorff space \( Y \), then \( f \) is a homeomorphism.
   (b) Give an example of topological spaces \( X \) and \( Y \) and a continuous bijection \( f: X \to Y \) that is not a homeomorphism.