1. (*) (a) Prove that a non-empty connected subset of a topological space $X$ that is both open and closed is a connected component of $X$.

(b) In general, is a connected component of a topological space open? Is it necessarily closed? Justify your answers.

2) (*) Let $X$ be a simply ordered set (i.e., given $x \neq y$ in $X$, either $x < y$ or $y < x$). Consider $X$ as a topological space, with the corresponding order topology. Show that $X$ is Hausdorff.

(b) Give an example of a compact topological space which is $T_1$ (i.e., one-point sets are closed) but not Hausdorff.

3) Let $f : [0, 1] \to [0, 1]$ be continuous (where $[0, 1]$ is regarded as a subspace of the real line $\mathbb{R}$, with the usual topology.) Show that there is a point $x \in [0, 1]$ such that $f(x) = x$.

4) Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point. Show that your example works.

5) Show that a connected metric space with more than one point must be uncountable.

6) (a) Let $X$ be a metric space. Suppose that for some $\epsilon > 0$, every $\epsilon$-ball in $X$ has compact closure. Show that $X$ is complete.

(b) Suppose that for each $x \in X$ there is an $\epsilon_x$ (depending on $x$) such that the ball $B(x, \epsilon_x)$ has compact closure. Show by means of an example that $X$ need not be complete.