Topology I Comprehensive Exam. January, 2002.

Instructions: Do the two starred problems. Then, do two out of the remaining four problems. Hand in a total of four problems. You have 2 and 1/2 hours for this test. Good luck!

1. (*) (a) Prove that a non-empty connected subset of a topological space X that is both open and closed is a connected component of X.

(b) In general, is a connected component of a topological space open? Is it necessarily closed? Justify your answers.

2) (*) Let X be a simply ordered set (i.e., given $x \neq y$ in X, either $x < y$ or $y < x$). Consider X as a topological space, with the corresponding order topology. Show that X is Hausdorff.

(b) Give an example of a compact topological space which is T_1 (i.e., one-point sets are closed) but not Hausdorff.

3) Let $f : [0,1] \to [0,1]$ be continuous (where $[0,1]$ is regarded as a subspace of the real line $\mathbf{R}$, with the usual topology.) Show that there is a point $x \in [0,1]$ such that $f(x) = x$.

4) Find a function $f : \mathbf{R} \to \mathbf{R}$ that is continuous at precisely one point. Show that your example works.

5) Show that a connected metric space with more than one point must be uncountable.

6) (a) Let X be a metric space. Suppose that for some $\epsilon > 0$, every ϵ-ball in X has compact closure. Show that X is complete.

(b) Suppose that for each $x \in X$ there is an ϵ_x (depending on x) such that the ball $B(x, \epsilon_x)$ has compact closure. Show by means of an example that X need not be complete.