MATH 7512 TOPOLOGY II CORE II EXAM AUGUST 2000

All spaces that arise on this exam are path-connected and locally path-connected. You have 3 and 1/2 hours to complete this test.

Do **any six** of the following problems. Indicate clearly which problems you do.

1. Let X be a path-connected topological space, and let x and y be two points in X. Show that the fundamental groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.

2. State the Seifert-Van Kampen Theorem. Use this theorem to show that the n-sphere $\mathbb{S}^n$ is simply connected for $n \geq 2$.

3. Let $p : \widetilde{X} \to X$ be a covering space, with $p^{-1}(x)$ finite for all $x \in X$. Show that $\widetilde{X}$ is Hausdorff if and only if X is Hausdorff.

4. Let A be a subspace of X, and $x_0 \in A$. If $A \subset X$ is a retract, and $i : A \hookrightarrow X$ is the inclusion map, prove that the induced homomorphism $i_* : \pi(A, x_0) \to \pi(X, x_0)$ on fundamental groups is injective.

5. Let $p : \widetilde{X} \to X$ be a covering space, and let x and y be two points in X. Show that the sets $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality.

6. Determine all isomorphism classes of covering spaces of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$. Exhibit an explicit covering space in each isomorphism class.

7. Let $X = \mathbb{S}^1 \bigvee \mathbb{S}^1$ be a bouquet of two circles. Exhibit a two-fold covering space $p : \widetilde{X} \to X$. Choose basepoints $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, and determine the homomorphism on fundamental groups $p_* : \pi(\widetilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ induced by the map p as explicitly as possible. Is the homomorphism p_* injective? Explain.

8. Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.