Do **any five** of the following problems. Indicate clearly which five problems you have selected. You have three and a half hours for the test. Good luck!

The theta-space referred to in problems 1 and 2 is the union of the unit circle in $\mathbb{R}^2$ and the interval $[-1, 1]$ on the x-axis.

1. Let X be the theta-space and let $Y = S^1 \vee S^1$, the wedge of two circles. Show that X and Y are homotopy equivalent, but not homeomorphic.

2. State the Seifert-van Kampen Theorem, and use it to find the fundamental group of the theta-space.

3. Let A be a subspace of X, let $a_0 \in A$, and let $i\colon A \to X$ be the inclusion map. Prove that if A is a retract of X then the induced homomorphism $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective, and deduce that S^1 is not a retract of B^2.

4. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

5. Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, where E is path-connected. Let $H = p_*(\pi_1(E, e_0))$ be the image of the fundamental group of E in $\pi_1(B, b_0)$. Prove that there is a bijection from the set $\pi_1(B, b_0)/H$ of right cosets of H to the fiber $p^{-1}(b_0)$.

6. Let $X = S^1 \vee S^1$, the wedge of two circles. Exhibit both a regular and an irregular 3-fold covering of X. Does X have an irregular 2-fold covering? Explain.

7. Let X be a path-connected and locally path-connected space. State a necessary and sufficient condition for X to have a universal covering space. Give an example of a path-connected and locally path-connected space that does **not** have a universal covering space, and prove directly that it does not satisfy the condition.