All spaces that arise on this exam are path-connected and locally path-connected.

If X is a topological space, and $x_0 \in X$ is a point in X, the fundamental group of X based at x_0 is denoted by $\pi(X, x_0)$.

Do **any five** of the following problems. Indicate clearly which problems you do.

1. Recall that a (non-empty) subset X of $\mathbb{R}^n$ is *convex* if the line segment between any two points in X is contained in X. If X is convex, show that X is simply connected.

2. State the Seifert-Van Kampen Theorem. Use this theorem to find the fundamental group of $X = \mathbb{S}^1 \bigvee \mathbb{S}^1$, a bouquet of two circles.

3. Let $p : \widetilde{X} \to X$ be a covering space. If X is simply connected, show that p is a homeomorphism of $\widetilde{X}$ onto X.

4. Suppose that X and Y are topological spaces, and let $x_0 \in X$ and $y_0 \in Y$. Prove that $\pi(X \times Y, (x_0, y_0)) \cong \pi(X, x_0) \times \pi(Y, y_0)$.

5. Let $p : \widetilde{X} \to X$ be a covering space (with $\widetilde{X}$ path-connected). For $x \in X$, there is a right action of $\pi(X, x)$ on the fiber $p^{-1}(x)$. Define this action, show that it is a group action, and show that this action is transitive.

6. Let A be a subspace of X, let $x_0 \in A$, and let $i : A \hookrightarrow X$ be the inclusion map.

(a) Give an example where the induced homomorphism $i_* : \pi(A, x_0) \to \pi(X, x_0)$ is not injective. Explain.

(b) Give a topological condition on A and X which insures that the homomorphism $i_* : \pi(A, x_0) \to \pi(X, x_0)$ is injective. Explain.

7. Determine all isomorphism classes of covering spaces of the circle $\mathbb{S}^1$. Exhibit an explicit covering space in each isomorphism class.