Do **any five** of the following problems. Indicate clearly which problems you do. You have 3 and 1/2 hours for this test. Good luck!

1. A subset X of $\mathbb{R}^n$ is *convex* if the line segment between any two points of X is contained in X; it is *star-convex* if there is some point x_0 in X such that the line segment from x_0 to any other point of X is contained in X. Give an example of a star-convex set that is not convex, and prove that any star-convex set is contractible.

2. Let $f: X \to Y$ be continuous. Let x_1 and x_2 be points of X, and let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. There are induced homomorphisms $f_{1*}: \pi_1(X, x_1) \to \pi_1(Y, y_1)$ and $f_{2*}: \pi_1(X, x_2) \to \pi_1(Y, y_2)$. Show that if X is path-connected then there are isomorphisms $\phi: \pi_1(X, x_1) \to \pi_1(X, x_2)$ and $\psi: \pi_1(Y, y_1) \to \pi_1(Y, y_2)$ such that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{f_{1*}} & \pi_1(Y, y_1) \\ \downarrow \phi & & \downarrow \psi \\ \pi_1(X, x_2) & \xrightarrow{f_{2*}} & \pi_1(Y, y_2) \end{array}$$

3. Let X be the subspace of $\mathbb{R}^2$ that is the union of two circles of radius 1 centered at $(-2, 0)$ and $(2, 0)$ and the line segment from $(-1, 0)$ to $(1, 0)$. State the Seifert-van Kampen Theorem, and use it to find the fundamental group of X.

4. Complete the following definition: topological spaces X and Y are *homotopy equivalent* if .... Prove that homotopy equivalence is an equivalence relation.

5. Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map, where X is path-connected and locally path-connected, and $\tilde{X}$ is simply-connected. (That is, $\tilde{X}$ is the universal covering space of X). Prove that the group of covering transformations of $\tilde{X}$ is isomorphic to the fundamental group of X.

6. Let X be a path-connected and locally path-connected space.

(a) State a necessary and sufficient condition for X to have a universal covering space.

(b) Assume that the condition of part (a) holds, and let $\tilde{X}$ be any path-connected covering space of X. Define what it means for $\tilde{X}$ to be a regular covering space, and prove that if $\tilde{X}$ is a two-fold covering space then it is regular.

7. Let X be the space described in problem 3. Exhibit both a regular and an irregular 3-fold covering of X.