Applications of vector-valued modular forms

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Applications of vector-valued modular forms

- Let $\Gamma(1) = \mathrm{PSL}_2(\mathbf{Z})$
- Write

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

- $\Gamma(1)$ has a presentation

$$\Gamma(1) = \langle R, S \mid R^3,\ S^2 \rangle.$$

- In particular, $\Gamma(1)$ is a quotient of the free nonabelian group on two generators

- Let $\rho\colon \Gamma(1) \to \mathrm{GL}_n(\mathbf{C})$ be a complex representation of $\Gamma(1)$
- Let k be an integer.
- Let $\mathfrak{H} = \{\tau \in \mathbf{C} \mid \Im\tau > 0\}$ denote the upper half plane.

Definition

A *vector-valued modular function* of weight k with respect to ρ is a holomorphic function $F\colon \mathfrak{H} \to \mathbf{C}^n$ such that

$$F(\gamma\tau) = \rho(\gamma)(c\tau + d)^k F(\tau) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1),$$

and such that F satisfies a "condition at infinity" (explained on next slide)

- If F is vector-valued modular for a rep. ρ,

$$\implies F(\tau+1) = F(T\tau) = \rho(T)F(\tau) \quad \text{for all} \quad \tau \in \mathcal{H}\,.$$

- Matrix exponential surjective, $\therefore$ $\rho(T) = e^{2\pi i L}$ for some matrix L (not unique).
- Then $\tilde{F}(\tau) = e^{-2\pi i L\tau}F(\tau)$ satisfies

$$\tilde{F}(\tau+1) = e^{-2\pi i L\tau} e^{-2\pi i L} \rho(T) F(\tau) = \tilde{F}(\tau).$$

- Meromorphy condition at infinity: *insist $\tilde{F}$ has a left finite Fourier expansion for all choices of logarithm L.*
- Can use Deligne's canonical compactification of a vector bundle with a regular connection on a punctured sphere to define holomorphic forms in a natural way.

Example:

- Let ρ denote the trivial representation
- Then: vector-valued forms are scalar forms of level 1
- Two examples are

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n, \qquad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

- The ring generated by the (holomorphic) forms of level 1 in all (integer) weights is

$$\mathbf{C}[E_4, E_6].$$

Example:

- More generally let ρ be a 1-dim rep of $\Gamma(1)$
- ρ factors through abelianization of $\Gamma(1)$, which is $\mathbf{Z}/6\mathbf{Z}$
- Let χ be the character of $\Gamma(1)$ such that $\chi(T) = e^{2\pi i/6}$. Then $\rho = \chi^r$ for some $0 \leq r < 6$.
- The $\mathbf{C}[E_4, E_6]$-module generated by vvmfs of all weights for χ^r is free of rank 1:

$$\mathcal{H}(\chi^r) = \mathbf{C}[E_4, E_6]\eta^{4r},$$

where η is the *Dedekind η-function*

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

Example of q-expansion condition:

- η^2 is a vvmf for a character χ with $\chi(T) = e^{2\pi i/6}$.
- Possible choices of exponent are $\frac{1}{6} + m$ for $m \in \mathbf{Z}$.
- The corresponding q-expansion is

$$\tilde{\eta}^2(q) = q^{-m} \prod_{n \geq 1} (1 - q^n)^2.$$

- Deligne's canonical compactification corresponds to taking $m = 0$.

Another example of q-expansion condition

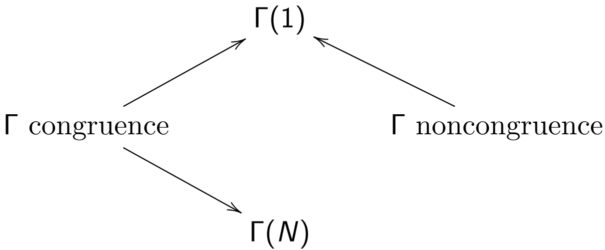
- Let ρ be $M_k(\Gamma(N))$ for some $N \geq 1$
- Elements $f \in M_k(\Gamma(N))$ don't have well-defined q-expansion: if $f(q_N) = \sum_{n\geq 0} a_n q_N^n$ then T stabilizes infinity, but changes the q_N-expansion:

$$(f|T)(q_N) = \sum_{n\geq 0} (a_n \zeta_N^n) q_N^n$$

- Suppose can find basis such that $\rho(T) = \mathrm{diag}(\zeta_{n_1}, \ldots, \zeta_{n_r})$, where $n_i \mid N$
- Basis elements then have form $f(q_N) = q_N^{N/n_i} \sum_{n\geq 0} a_n q^n$ and the q-expansion is $\sum_{n\geq 0} a_n q^n$.

Vector-valued modular forms and noncongruence modular forms

- a subgroup $\Gamma \subseteq \Gamma(1)$ is noncongruence if it's of finite index and does not contain $\Gamma(N)$ for any N.
- Most subgroups of $\Gamma(1)$ of finite index are noncongruence
- Idea of Selberg to study noncongruence forms: can't go down to $\Gamma(N)$, but it's a finite distance up to $\Gamma(1)$.
- Go up by using vector-valued modular forms



Applications of vector-valued modular forms

The following *Free-module theorem* is very useful:

Theorem (Marks-Mason, Knopp-Mason, Bantay-Gannon)

Let ρ denote an n dimensional complex representation of Γ. Let $\mathcal{H}(\rho)$ denote the $\mathbf{C}[E_4, E_6]$-module generated by all vvmfs of varying weight. Then $\mathcal{H}(\rho)$ is free of rank n as a $\mathbf{C}[E_4, E_6]$-module.

Note: we stated this previously for 1-dim reps!

Example: two-dimensional irreducibles

- Let ρ be a 2-dim irrep
- $\rho(T)$ must have distinct eigenvalues, otherwise ρ factors through abelianization of Γ
- Assume that $\rho(T)$ is diagonal and of finite order (to avoid introducing logarithmic terms), and write

$$\rho(T) = \begin{pmatrix} e^{2\pi i r_1} & 0 \\ 0 & e^{2\pi i r_2} \end{pmatrix}$$

  with $r_1, r_2 \in [0, 1)$.
- Let $\mathcal{H}(\rho)$ denote the $\mathbf{C}[E_4, E_6]$-module of vector-valued modular forms for ρ.

Theorem (F-Mason, 2013)

Let notation be as on the previous slide, and let $K = 1728/j$ where j is the usual j-function. Then

$$\mathcal{H}(\rho) = \mathbf{C}[E_4, E_6]F \oplus \mathbf{C}[E_4, E_6]DF$$

where:

$$F = \eta^{2k} \begin{pmatrix} K^{\frac{6(r_1-r_2)+1}{12}} {}_2F_1\left(\frac{6(r_1-r_2)+1}{12}, \frac{6(r_1-r_2)+5}{12}; r_1 - r_2 + 1; K\right) \\ K^{\frac{6(r_2-r_1)+1}{12}} {}_2F_1\left(\frac{6(r_2-r_1)+1}{12}, \frac{6(r_2-r_1)+5}{12}; r_2 - r_1 + 1; K\right) \end{pmatrix},$$

$$k = 6(r_1 + r_2) - 1,$$

$$D = q\frac{d}{dq} - \frac{k}{12}E_2.$$

Idea of proof:

- By free-module theorem can write $\mathcal{H}(\rho) = \langle F, G \rangle$ for two vvmfs F and G
- WLOG assume weight $F \leq$ weight G
- Then $DF = \alpha F + \beta G$ for modular forms α and β
- But α must be of weight 2, hence $\alpha = 0$ and $DF = \beta G$.
- If $\beta = 0$ then $DF = 0$ and coordinates of F must be multiples of a power of η
- But then $\Gamma(1)$ acts by a scalar on F, and can use this to contradict the irreducibility of ρ
- Hence $DF = \beta G$, and by weight considerations β is nonzero scalar
- So: we can replace G by DF.

Continuation of proof:

- Thus we've shown that $\mathcal{H}(\rho) = \langle F, DF \rangle$ for some vvmf F of minimal weight.
- Can write $D^2 F = \alpha E_4 F$ for a scalar α.
- If weight of F is zero, this is the pullback of a hypergeometric differential equation on $\mathbf{P}^1 - \{0, 1, \infty\}$ via $K = 1728/j$
- Can reduce to weight 0 case by dividing by a power of η, since $D(\eta) = 0$

Example: three-dimensional irreducibles

- Let ρ be a 3-dim irrep
- Again, $\rho(T)$ must have distinct eigenvalues
- Assume that $\rho(T)$ is diagonal and of finite order (to avoid introducing logarithmic terms), and write

$$\rho(T) = \mathrm{diag}(e^{2\pi i r_1}, e^{2\pi i r_2}, e^{2\pi i r_3}).$$

with $r_1, r_2, r_3 \in [0, 1)$.

Theorem (F-Mason, 2013)

Let notation be as on the previous slide. Then

$$\mathcal{H}(\rho) = \mathbf{C}[E_4, E_6]F \oplus \mathbf{C}[E_4, E_6]DF \oplus \mathbf{C}[E_4, E_6]D^2F$$

where:

$$F = \eta^{2k} \begin{pmatrix} K^{\frac{a_1+1}{6}}\,{}_3F_2\left(\frac{a_1+1}{6}, \frac{a_1+3}{6}, \frac{a_1+5}{6}; r_1 - r_2 + 1, r_1 - r_3 + 1; K\right) \\ K^{\frac{a_2+1}{6}}\,{}_3F_2\left(\frac{a_2+1}{6}, \frac{a_2+3}{6}, \frac{a_2+5}{6}; r_2 - r_1 + 1, r_2 - r_3 + 1; K\right) \\ K^{\frac{a_3+1}{6}}\,{}_3F_2\left(\frac{a_3+1}{6}, \frac{a_3+3}{6}, \frac{a_3+5}{6}; r_3 - r_2 + 1, r_3 - r_1 + 1; K\right) \end{pmatrix},$$

$$k = 4(r_1 + r_2 + r_3) - 2,$$

and for $\{i, j, k\} = \{1, 2, 3\}$ *we write* $a_i = 4r_i - 2r_j - 2r_k$.

- We used our results on 2-dim vvmfs to verify the unbounded denominator conjecture in those cases
- Unfortunately, no noncongruence examples arise there!
- 3-dim case: infinitely many noncongruence examples
- Results of next section were motivated by the question: can we use our results to prove things about noncongruence modular forms?

Applications of vector-valued modular forms

Representations of $\Gamma(1) = \mathrm{PSL}_2(\mathbf{Z})$:

- $\Gamma(1)$ is discrete and its irreps of fixed dimension are parameterized by an algebraic variety (character variety)
- Most irreps are of infinite image and the corresponding vvmfs are weird (the compoments are modular with respect to a thin subgroup of $\Gamma(1)$)
- We'll focus on reps with *finite image*
- Equivalently: we consider irreps ρ with $\ker \rho$ a finite index subgroup of $\Gamma(1)$
- Components of vvmfs for ρ are then scalar forms for $\ker \rho$

Representations of $\Gamma(1)$ of finite image:

- Finite image reps come in two flavours: primitive and imprimitive
- Imprimitive means it's induced from a nontrivial subgroup
- Primitive means it's not
- There are *finitely many* primitives of each dimension
- In dimension 3, all primitives with finite image have congruence kernel
- we'd thus like to classify the (infinitely many) 3-dimensional imprimitive representations of $\Gamma(1)$ with finite image.
- all but finitely many of these imprimitive ρ have a noncongruence subgroup as kernel.

Three-dimensional imprimitive irreps of $\Gamma(1)$ of finite image:

- A 3-dim imprimitive is induced from an index-3 subgroup

Lemma

$\Gamma(1)$ *contains exactly* 4 *subgroups of index* 3*. One is a normal congruence subgroup of level* 3*, while the others are conjugate with* $\Gamma_0(2)$*.*

- The normal subgroup has finite abelianization and gives rise to a finite number of congruence representations
- The other index 3 subgroups have infinite abelianization and many characters
- Since they're conjugate, we can assume we're inducing a character from $\Gamma_0(2)$.

Characters of $\Gamma_0(2)$

- Let

$$U := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad V := TU^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}.$$

Then $\Gamma_0(2) = \langle T, U \rangle = \langle T^2, U \rangle \rtimes \langle V \rangle$ and

$$\Gamma_0(2)/\Gamma_0(2)' \cong \mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})$$

- U generates the copy of $\mathbf{Z}$ and V generates $\mathbf{Z}/2\mathbf{Z}$
- Thus $\chi \colon \Gamma_0(2) \to \mathbf{C}^\times$ with finite image is classified by data

$$\chi(U) = \lambda$$
$$\chi(V) = \varepsilon$$

where $\lambda^n = 1$ for some $n \geq 1$ and $\varepsilon^2 = 1$.

The representation $\rho = \mathrm{Ind}_{\Gamma_0(2)}^{\Gamma(1)}(\chi)$:

- Let $\chi\colon \Gamma_0(2) \to \mathbf{C}^\times$ be a finite order character, with $\chi(U) = \lambda$ and $\chi(V) = \varepsilon$.
- If $\rho = \mathrm{Ind}_{\Gamma_0(2)}^{\Gamma(1)}(\chi)$, one checks that $\rho(T)$ has eigenvalues $\{\varepsilon\lambda, \sigma, -\sigma\}$ where $\sigma^2 = \bar{\lambda}$.
- Further, one can prove the following.

Proposition (F-Mason, 2014)

Let n be the order of the root of unity $\lambda = \chi(U)$. Then the following hold:

1. *ρ is irreducible if and only if $n \nmid 3$;*
2. *$\ker \rho$ is a congruence subgroup if and only if $n \mid 24$.*

Thus: previous formulae describe an infinite collection of noncongruence modular forms in terms of η, j and ${}_3F_2$

Proposition

Let $\chi\colon \Gamma_0(2) \to \mathbf{C}^\times$ *denote a character of finite order. Let* n *be the order of the primitive* n*th root of unity* $\chi(U)$*, and assume that* $n \nmid 3$*. Let* $\rho\colon \Gamma_0(2) \to \mathrm{GL}_3(\mathbf{C})$ *denote a representation that is isomorphic with* $\mathrm{Ind}_{\Gamma_0(2)}^{\Gamma(1)} \chi$*, and which satisfies*

$$\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mathrm{diag}(e^{2\pi i r_1}, e^{2\pi i r_2}, e^{2\pi i r_3})$$

where $r_1, r_2, r_3 \in [0,1)$*. Let* $\mathcal{H}(\rho)$ *denote the graded module of vector-valued modular forms for* ρ*, and let* $M(\Gamma_0(2),\chi)$ *denote the graded module of scalar-valued modular forms on* $\ker\chi$ *that transform via the character* χ *under the action of* $\Gamma_0(2)$*. Then, after possibly reordering the coordinates, projection to the first coordinate defines an isomorphism* $\mathcal{H}(\rho) \cong M(\Gamma_0(2),\chi)$ *of graded* $\mathbf{C}[E_4, E_6]$*-modules.*

Idea of proof:

- WLOG reorder the exponents r_i so that the first coordinate of $F \in \mathcal{H}(\rho)$ lives in $M(\Gamma_0(2), \chi)$.
- Let γ_1, γ_2 and γ_3 denote distinct coset representatives of $\Gamma_0(2)$ in $\Gamma(1)$ with $\gamma_1 = 1$.
- Given $g \in M(\Gamma_0(2), \chi)$, consider the vector function $F = (g|\gamma_1, g|\gamma_2, g|\gamma_3)^T$.
- Then $F \in \mathcal{H}(\rho)$ and its first coordinate is g, so this gives an inverse to the projection map.

- This gives an infinite collection of noncongruence modular forms that are described in terms of hypergeometric series
- Note that if $f \in M_k(\Gamma_0(2), \chi)$, then $f^{2n} \in M_{2kn}(\Gamma_0(2))$ is a congruence modular form, so in a sense these examples are rather elementary
- We've used these results to prove congruences and unbounded denominator type results for these vector-valued modular forms
- In the remainder of the talk we wish to describe some computations with CM values of these noncongruence modular forms

Applications of vector-valued modular forms

- Fix an imag. quadratic field $E/\mathbf{Q}$ and an embedding $E \subseteq \mathbf{C}$
- Let δ_k denote the Maass-Shimura operator

$$\delta_k(f)(\tau) = \frac{1}{2\pi i}\left(\frac{df}{d\tau}(\tau) + \frac{kf(\tau)}{z - \bar{z}}\right).$$

- Let δ_k^r denote the rth iterate of δ_k
- Recall that δ_k commutes with the slash operator

Theorem (Shimura)

There exists a complex period $\Omega_E \in \mathbf{C}^\times$ such that for all congruence modular forms f with algebraic Fourier coefficients, for all $\tau \in \mathcal{H} \cap E$, and for all integers $r \geq 0$, one has

$$\frac{\delta_k^r f(\tau)}{\Omega_E^{k+2r}} \in \bar{\mathbf{Q}},$$

where k is the weight of f.

Actually, Shimura says much more about the arithmetic nature of these values, and that is the hard part of his paper, but we'll ignore this for now.

- We (and probably many other mathematicians) have observed that Shimura's result extends to noncongruence modular forms
- Basic idea: reduce to weight 0 by dividing by a power of η
- Then the weight 0 form lies in a finite extension of $\mathbf{C}(j)$, so it has a minimal polynomial in $\mathbf{C}[X, j]$.
- If the form f has algebraic Fourier coefficients, can find a minimal polynomial $P(X, j) \in \bar{\mathbf{Q}}[X, j]$.
- But then $P(f(\tau), j(\tau)) = 0$. If $\tau \in E \cap \mathcal{H}$, then $j(\tau)$ is algebraic, and this shows that $f(\tau)$ is also algebraic

- The arithmetic nature of noncongruence CM-values is a mystery.
- Could they describe nonabelian extensions of quadratic imaginary extensions?
- Some evidence:
- Nonabelian extensions outnumber abelian ones, just like noncongruence groups outnumber congruence ones
- There is a history of finding roots of general polynomials using special functions: e.g.

$$-a\ {}_4F_3\left(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\frac{1}{2},\frac{3}{4},\frac{5}{4};-5\left(\frac{5a}{4}\right)^4\right)$$

is a root of x^5+x+a.

Example. The rational j-invariants

- Let F denote the form

$$F = K^{-\frac{1}{15}}\ {}_3F_2\left(-\frac{1}{15}, \frac{4}{15}, \frac{3}{5}; \frac{9}{10}, \frac{2}{5}; K\right)$$

  where $K = 1728/j$.

- This is a noncongruence form of weight 0 on $\Gamma_0(2)$ with a character of order 10. It's defined over $\mathbf{Q}(\zeta_5)$.
- The form $6F(j)$ satisfies the equation $Q(6F(j), j) = 0$ where

$$Q(X,j) = X^{45} + (2^8 \cdot 3 - j) \cdot 2^9 \cdot 3^{12} \cdot X^{30} + 2^{34} \cdot 3^{25} \cdot X^{15} + 2^{51} \cdot 3^{36}$$

-Disc	Minimal polynomial of $6F(j)$
3	$X^{15} + 2^{17}3^{12}$
$3 \cdot 2^2$	$X^{30} - 2^{21}3^{12}13X^{15} + 2^{38}3^{24}$
$3 \cdot 3^2$	$X^{45} + 2^{17}3^{13}16001^1X^{30} + 2^{34}3^{25}X^{15} + 2^{51}3^{36}$
4	$X^5 - 2^63^4$
$4 \cdot 2^2$	$X^{10} - 2^93^4X^5 - 2^{13}3^8$
7	$X^{10} - 2^73^4X^5 + 2^{14}3^8$
$7 \cdot 2^2$	$X^{10} - 2^{11}3^4X^5 + 2^{14}3^8$
8	$X^{10} - 2^73^4X^5 - 2^{12}3^8$
11	$X^{30} - 2^83^4X^{25} + 2^{16}3^8X^{20} + 2^{18}3^{12}X^{15} - 2^{25}3^{16}X^{10} + 2^{34}3^{24}$
19	$X^{30} - 2^83^5X^{25} + 2^{16}3^{10}X^{20} + 2^{18}3^{12}X^{15} - 2^{25}3^{17}X^{10} + 2^{34}3^{24}$
43	$X^{30} - 2^93^55X^{25} + 2^{18}3^{10}5^2X^{20} + 2^{18}3^{12}X^{15} - 2^{26}3^{17}5X^{10} + 2^3$
67	$X^{30} - 2^83^55^111X^{25} + 2^{16}3^{10}5^211^2X^{20} + 2^{18}3^{12}X^{15} - 2^{25}3^{17}5^11$
163	$X^{30} - 2^93^55^123^129X^{25} + 2^{18}3^{10}5^223^229^2X^{20} + 2^{18}3^{12}X^{15} - 2^{26}$

- In the case of disc. $-D$, let $B = \mathbf{Q}(\sqrt{-D}, \zeta_5)$.
- Then a root of the min poly above generates an abelian Galois extension of B
- In all cases except the case when $D = 3 \cdot 3^2$, the Galois group is in fact cyclic
- Note that in this case F^{10} is a congruence form of weight 0, and that explains why one observes Kummer extensions in studying these number fields
- It would be exciting to compute a similar example using a primitive representation of $\mathrm{PSL}_2(\mathbf{Z})$ with noncongruence kernel!
- Will one observe nonabelian extensions?

Thanks for listening!