Hypergeometric Series and Gaussian Hypergeometric Functions

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$_2F_1$-Hypergeometric Series

- Let $a, b, c \in \mathbb{R}$. The hypergeometric function $_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ is defined by

$$_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,$$

  where $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol.
- Euler's integral representation of the $_2F_1$ with $c > b > 0$

$$_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] = \frac{1}{B(b, c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx,$$

  where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

  is the Beta function.

Hypergeometric Functions over Finite Fields

Let $q = p^s$ be a prime power. Let $\widehat{\mathbb{F}_q^\times}$ denote the group of multiplicative characters on $\mathbb{F}_q^\times$. Extend $\chi \in \widehat{\mathbb{F}_q^\times}$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$.

Gaussian Hypergeometric Function. (Greene, 1984) Let $\lambda \in \mathbb{F}_q$, and $A, B, C \in \widehat{\mathbb{F}_q^\times}$.

- $${}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda\Big)_q := \varepsilon(\lambda)\frac{BC(-1)}{q}\sum_{x\in\mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x),$$
  where ε is the trivial character.
- $${}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda\Big)_q := \frac{q}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(\lambda),$$
  where $\binom{A}{B} := \frac{B(-1)}{q}J(A,\overline{B})$ is the normalized Jacobi sum of A, B.

Legendre Family

For $\lambda \neq 0, 1$, let

$$E_\lambda : \ y^2 = x(x-1)(x-\lambda)$$

be the elliptic curve in Legendre normal form.

- The periods of the Legendre family of elliptic curves are

$$\Omega(E_\lambda) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

- If $0 < \lambda < 1$, then

$${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right] = \frac{\Omega(E_\lambda)}{\pi}.$$

If $\lambda \in \mathbb{Q}$, and E_λ has good reduction at prime p, we can express $\#E_\lambda(\mathbb{F}_p)$ in terms of Gaussian hypergeometric functions.

Legendre Family over Finite Fields

Legendre family of elliptic curves over $\mathbb{F}_p$:

$$\widetilde{E_\lambda}: \; y^2 = x(x-1)(x-\lambda)$$

Trace of Frobenius:

$$a_p(\lambda) = p + 1 - \#\widetilde{E_\lambda}(\mathbb{F}_p), \quad \lambda \neq 0, 1$$

Koike 1992.
If p is an odd prime, then

$$p \, {}_2F_1 \left[\begin{matrix} \eta_2 & \eta_2 \\ & \varepsilon \end{matrix} ; \lambda \right]_p = -\eta_2(-1) a_p(\lambda), \quad \lambda \neq 0, 1,$$

where ε is the trivial character and η_2 is the quadratic character.

$$E_\lambda : \ y^2 = x(x-1)(x-\lambda), \quad \lambda \in \mathbb{Q} - \{0, 1\}.$$

- If $0 < \lambda < 1$, then

$$_2F_1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{bmatrix}; \lambda \Bigg] = \frac{\Omega(E_\lambda)}{\pi} = \frac{\Omega(E_\lambda)}{\Gamma(\frac{1}{2})^2}.$$

- If p is an odd prime with $\mathrm{ord}_p(\lambda(\lambda-1)) = 0$, then

$$_2F_1 \left[\begin{matrix} \eta_2 & \eta_2 \\ & \varepsilon \end{matrix}; \lambda \right]_p = -\frac{a_p(\lambda)}{p\eta_2(-1)} = \frac{-a_p(\lambda)}{g(\eta_2)^2}.$$

- If $\lambda = \frac{1}{2}$, $p \equiv 1 \mod 4$, we have

$$\frac{\sqrt{2}}{2\pi}\Omega(E_\lambda) = \mathrm{Re}\binom{1/4}{1/2}, \quad \frac{-\eta_2(2)}{2p}a_p(\lambda) = \mathrm{Re}\binom{\eta_4}{\eta_2},$$

where η_4 is a character of order 4.

For $m \in \mathbb{Z}^+$, define the truncated $_2F_1$-hypergeometric series by

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \lambda\right]_m := \sum_{k=0}^{m} \frac{(a)_k(b)_k}{(c_k)k!}\lambda^k. $$

When $a_p(\lambda)$ is not divisible by p, Dwork shows that

$$ f_p(\lambda) := \lim_{s\to\infty} {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}; \hat{\lambda}\right]_{p^s-1} \Big/ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}; \hat{\lambda}\right]_{p^{s-1}-1} $$

is the unit root of $T^2 - a_p(\lambda)T + p$, where $\hat{\lambda}$ is the image of λ under the Teichmüller character.

Example. When $\lambda = -1$, $p \equiv 1 \pmod 4$,

- $-a_p(-1) = p \cdot {}_2F_1\left(\begin{matrix} \eta_2 & \eta_2 \\ & \varepsilon \end{matrix}; -1\right)_p = J(\eta_4, \eta_2) + J(\overline{\eta_4}, \eta_2).$
- $f_p(\lambda) = \frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)}{\Gamma_p\left(\frac{3}{4}\right)}$, where $\Gamma_p(\cdot)$ is the p-adic Gamma function.

Motivations

hypergeometric series	$\longleftrightarrow$	periods
Gaussian hypergeometric series	$\longleftrightarrow$	Golais representations
truncated hypergeometric series	$\longleftrightarrow$	unit roots

Motivation

Investigate the relationships among hypergeometric series, truncated hypergeometric series, and Gaussian hypergeometric functions through some families of hypergeometric algebraic varieties.

- $y^N = x^i(1-x)^j(1-\lambda x)^k$
- $y^n = (x_1 x_2 \cdots x_{n-1})^{n-1}(1-x_1)\cdots(1-x_{n-1})(x_1 - \lambda x_2 x_3 \cdots x_{n-1})$

Generalized Legendre Curves

Let $N \geq 2$, and i, j, k be natural numbers with $1 \leq i, j, k < N$. For the smooth model X_λ of the curve

$$C_\lambda : \, y^N = x^i(1-x)^j(1-\lambda x)^k, \, \lambda \in \mathbb{Q} - \{0, 1\}$$

- a period can be chosen as

$$\mathcal{P}(\lambda) = B\left(1 - \frac{i}{N}, 1 - \frac{j}{N}\right) {}_2F_1 \left[\begin{matrix} \frac{k}{N} & \frac{N-i}{N} \\ & \frac{2N-i-j}{N} \end{matrix} ; \, \lambda \right],$$

- Let $\eta \in \widehat{\mathbb{F}_q^\times}$ be a character of order N. Then

$$\#X_\lambda(\mathbb{F}_q)\text{"} = \text{"}1 + q + q\sum_{m=1}^{N-1} \eta^{mj}(-1)\, {}_2F_1 \begin{pmatrix} \eta^{-km} & \eta^{im} \\ & \eta^{m(i+j)} \end{pmatrix} ; \lambda \Bigg)_q .$$

Generalized Hypergeometric Series/Functions

- For a positive integer n, and $\alpha_i, \beta_i \in \mathbb{C}$ with $\beta_i \notin \mathbb{Z}^-$, the hypergeometric series ${}_{n+1}F_n$ is defined by

$$ {}_{n+1}F_n\left[\begin{matrix}\alpha_0 & \alpha_1 & \ldots & \alpha_n \\ & \beta_1 & \ldots & \beta_n\end{matrix}; \lambda\right] := \sum_{k=0}^{\infty} \frac{(\alpha_0)_k}{(1)_k} \prod_{i=1}^{n} \frac{(\alpha_i)_k}{(\beta_i)_k} \cdot \lambda^k $$

where $(a)_0 := 1$, $(1)_k = k!$, and $(a)_k := a(a+1)\cdots(a+k-1)$.

- If n is a positive integer, and $A_i, B_i \in \widehat{\mathbb{F}_q^\times}$, then

$$ {}_{n+1}F_n\left(\begin{matrix}A_0 & A_1 & \ldots & A_n \\ & B_1 & \ldots & B_n\end{matrix}; \lambda\right)_q := \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A_0\chi}{\chi} \prod_{i=1}^{n} \binom{A_i\chi}{B_i\chi} \chi(\lambda). $$

Euler's Integral Formulae

When $\mathrm{Re}(\beta_\mathrm{r}) > \mathrm{Re}(\alpha_\mathrm{r}) > 0$,

$$
{}_{n+1}F_n\left[\begin{matrix}\alpha_0 & \alpha_1 & \dots & \alpha_n \\ & \beta_1 & \dots & \beta_n\end{matrix}; \ \lambda\right] =
$$

$$
\frac{\Gamma(\beta_n)}{\Gamma(\alpha_n)\Gamma(\beta_n-\alpha_n)}\int_0^1 x^{\alpha_n-1}(1-x)^{\beta_n-\alpha_n-1}\,{}_nF_{n-1}\left[\begin{matrix}\alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ & \beta_1 & \dots & \beta_{n-1}\end{matrix}; \ \lambda x\right]
$$

For characters $A_0, A_1, \dots, A_n, B_1, \dots, B_n$ in $\widehat{\mathbb{F}_q^\times}$,

$$
{}_{n+1}F_n\left(\begin{matrix}A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n\end{matrix}; \lambda\right)_q =
$$

$$
\frac{A_nB_n(-1)}{q}\cdot\sum_x A_n(x)\overline{A_n}B_n(1-x)\cdot{}_nF_{n-1}\left(\begin{matrix}A_0, & A_1, & \dots, & A_{n-1} \\ & B_1, & \dots, & B_{n-1}\end{matrix}; \lambda x\right)_q .
$$

Higher Dimensional Analogues of Legendre Curves

$C_{n,\lambda}: \quad y^n = (x_1x_2\cdots x_{n-1})^{n-1}(1-x_1)\cdots(1-x_{n-1})(x_1-\lambda x_2x_3\cdots x_{n-1})$

- $C_{2,\lambda}$ are known as Legendre curves.
- Up to a scalar multiple, ${}_nF_{n-1}\left[\begin{matrix} \frac{j}{n} & \frac{j}{n} & \cdots & \frac{j}{n} \\ & 1 & \cdots & 1 \end{matrix}; \lambda\right]$ for any $1 \le j \le n-1$, when convergent, can be realized as a period of $C_{n,\lambda}$.

Theorem (Deines, Long, Fuselier, Swisher, T.)

Let $q = p^e \equiv 1 \pmod{n}$ be a prime power. Let η_n be a primitive order n character and ε the trivial multiplicative character in $\widehat{\mathbb{F}_q^\times}$. Then

$$\#C_{n,\lambda}(\mathbb{F}_q) = 1 + q^{n-1} + q^{n-1}\sum_{i=1}^{n-1} {}_nF_{n-1}\left(\begin{matrix} \eta_n^i, & \eta_n^i, & \cdots, & \eta_n^i, \\ & \varepsilon, & \cdots, & \varepsilon, \end{matrix}; \lambda\right)_q.$$

Local L-functions of $C_{3,1}$ and $C_{4,1}$

Theorem (Deines, Long, Fuselier, Swisher, T.)

Let η_2, η_3, *and* η_4 *denote characters of order 2, 3, or 4, respectively, in* $\widehat{\mathbb{F}_q^\times}$.

- *Let* $q \equiv 1 \pmod 3$ *be a prime power. Then*

$$q^2 \cdot {}_3F_2\left(\begin{matrix}\eta_3, & \eta_3, & \eta_3 \\ & \varepsilon, & \varepsilon\end{matrix};1\right)_q = J(\eta_3,\eta_3)^2 - J(\eta_3^2,\eta_3^2).$$

- *Let* $q \equiv 1 \pmod 4$ *be a prime power. Then*

$$q^3 \cdot {}_4F_3\left(\begin{matrix}\eta_4, & \eta_4, & \eta_4, & \eta_4 \\ & \varepsilon, & \varepsilon, & \varepsilon\end{matrix};1\right)_q = J(\eta_4,\eta_2)^3 + qJ(\eta_4,\eta_2) - J(\overline{\eta_4},\eta_2)^2.$$

Ahlgren-Ono. For any odd prime p,

$$p^3 \cdot {}_4F_3\left(\begin{matrix}\eta_4^2, & \eta_4^2, & \eta_4^2, & \eta_4^2 \\ & \varepsilon, & \varepsilon, & \varepsilon\end{matrix}; 1\right)_p = -a(p) - p,$$

where $a(p)$ is the pth coefficient of the weight-4 Hecke eigenform $\eta(2z)^4\eta(4z)^4$, with $\eta(z)$ being the Dedekind eta function.

The factor of $Z_{C_{4,1}}$ corresponding to

$$y^2 = (x_1x_2x_3)^3(1-x_1)(1-x_2)(1-x_3)(x_1-x_2x_3)$$

is

$$Z_{C_{4,1}^{old}}(T,p) = \frac{(1-a(p)T+p^3T^2)(1-pT)}{(1-T)(1-p^3T)}.$$

- $q^3 \cdot {}_4F_3\left(\begin{matrix}\eta_4, & \eta_4, & \eta_4, & \eta_4 \\ & \varepsilon, & \varepsilon, & \varepsilon\end{matrix}; 1\right)_q = J(\eta_4, \eta_2)^3 + qJ(\eta_4, \eta_2) - J(\overline{\eta_4}, \eta_2)^2$
- Hasse-Davenport relation.
  Let $\mathbb{F}$ be a finite field and $\mathbb{F}_s$ an extension field over $\mathbb{F}$ of degree s. If $\chi \neq \varepsilon \in \widehat{F^\times}$ and $\chi_s = \chi \circ N_{\mathbb{F}_s/\mathbb{F}}$ a character of $\mathbb{F}_s$. Then

$$(-g(\chi))^s = -g(\chi_s).$$

The factor corresponding to new part is

$$(1 + (\beta_p^3 + \overline{\beta}_p^3)T + p^3T^2)\,(1 + (\beta_p + \overline{\beta_p})pT + p^3T^2)$$
$$(1 - (\beta_p^2 + \overline{\beta}_p^2)T + p^2T^2),$$

where $\beta_p = J(\eta_4, \eta_2)$.

Galois Representation corresponding to $Z_{C_{4,1}^{new}}(T, p)$

The Jacobi sum $J(\eta_4, \eta_2)$ can be viewed as the Hecke (or Grössencharacter) character ψ of $G_{\mathbb{Q}(\sqrt{-1})}$, which is corresponding to the elliptic curve with complex multiplication which has conductor 64.

By class field theory, ψ corresponds to a character χ of $G_{\mathbb{Q}(\sqrt{-1})}$. For each Frobenius class $\mathrm{Frob}_q \in G_{\mathbb{Q}(\sqrt{-1})}$ with $q \equiv 1 \pmod 4$,

$$-q^3 \cdot \sum_{i=1,3} {}_4F_3\begin{pmatrix} \eta_4^i, & \eta_4^i, & {\eta_4}^i, & \eta_4^i; 1 \\ & \varepsilon, & \varepsilon, & \varepsilon \end{pmatrix}_q$$

coincides with the trace of $\mathrm{Frob}_{\mathfrak{p}}$ under the 6-dimensional semisimple representation

$$\rho := \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{-1})}}^{G_{\mathbb{Q}}} \left(\overline{\chi}^3 \oplus (\overline{\chi}^2 \otimes \chi) \oplus \chi^2\right).$$

For $m \in \mathbb{Z}^+$, define

$$ {}_{n+1}F_n\left[\begin{matrix}\alpha_0 & \alpha_1 & \dots & \alpha_n \\ & \beta_1 & \dots & \beta_n\end{matrix}; \lambda\right]_m := \sum_{k=0}^{m} \frac{(\alpha_0)_k}{(1)_k} \prod_{i=1}^{n} \frac{(\alpha_i)_k}{(\beta_i)_k} \cdot \lambda^k. $$

Theorem (Deines, Long, Fuselier, Swisher, T.)

For each prime $p \equiv 1 \pmod 4$,

$$ {}_4F_3\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ & 1 & 1 & 1\end{matrix}; 1\right]_{p-1} \equiv (-1)^{\frac{p-1}{4}} \Gamma_p\left(\frac{1}{2}\right) \Gamma_p\left(\frac{1}{4}\right)^6 \pmod{p^4}. $$

Kilbourn.

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1\end{matrix}; 1\right]_{p-1} \equiv a(p) \pmod{p^3}. $$

Lemma. Let r, n, j be positive integers with $1 \le j < n$. Let $p \equiv 1 \pmod{n}$ be prime and $\eta_n \in \widehat{\mathbb{F}_p^\times}$ such that $\eta_n(x) \equiv x^{j(p-1)/n} \pmod{p}$ for each $x \in \mathbb{F}_p$. Then,

$$p^{r-1} \cdot {}_rF_{r-1}\left(\begin{matrix} \eta_n, & \eta_n, & \cdots, & \eta_n \\ & \varepsilon, & \cdots, & \varepsilon \end{matrix}; x\right)_p \equiv$$

$$(-1)^{r+1} \cdot {}_rF_{r-1}\left[\begin{matrix} \frac{n-j}{n} & \frac{n-j}{n} & \cdots & \frac{n-j}{n} \\ & 1 & \cdots & 1 \end{matrix}; \frac{1}{x}\right]_{(p-1)\left(\frac{n-j}{n}\right)}$$

$$+ (-1)^{r+1+\frac{(p-1)}{n}jr}\left(x^{(p-1)\frac{n-j}{n}} - x^{\frac{p-1}{n}j}\right) \pmod{p};$$

We have similar result for

$$p^{r-1} \cdot {}_rF_{r-1}\left(\begin{matrix} \overline{\eta_n}, & \overline{\eta_n}, & \cdots, & \overline{\eta_n} \\ & \varepsilon, & \cdots, & \varepsilon \end{matrix}; x\right)_p$$

Theorem (Deines, Long, Fuselier, Swisher, T.)

For $n \geq 3$, and $p \equiv 1 \pmod{n}$ prime,

$$\begin{aligned} {}_nF_{n-1}\left[\begin{matrix} \frac{n-1}{n} & \frac{n-1}{n} & \cdots & \frac{n-1}{n} \\ & 1 & \cdots & 1 \end{matrix} ; 1\right]_{p-1} &= \sum_{k=0}^{p-1} \binom{\frac{1-n}{n}}{k}^n (-1)^{kn} \\ &\equiv -\Gamma_p\left(\frac{1}{n}\right)^n \pmod{p^2}. \end{aligned}$$

Conjecture.

Let $n \geq 3$ be a positive integer, and p be prime such that $p \equiv 1 \pmod{n}$. Then

$${}_nF_{n-1}\left[\begin{matrix} \frac{n-1}{n} & \frac{n-1}{n} & \cdots & \frac{n-1}{n} \\ & 1 & \cdots & 1 \end{matrix} ; 1\right]_{p-1} \equiv -\Gamma_p\left(\frac{1}{n}\right)^n \pmod{p^3}.$$

p-adic Gamma Functions

Assume p is an odd prime.

Morita. The p-adic Gamma function $\Gamma_p : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^\times$ is the unique continuous function characterized by

$$\Gamma_p(n) = (-1)^n \prod_{0<i<n, p\nmid i} i, \; n \in \mathbb{Z}^+,$$

and

$$\Gamma_p(x) = \lim_{n\to x} \Gamma_p(n), \; x \in \mathbb{Z}_p.$$

Proposition.

- $\Gamma_p(0) = 1$
- $\Gamma_p(x+1)/\Gamma_p(x) = -x$ unless $x \in p\mathbb{Z}_p$ in which case the quotient takes value -1.
- $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$ where $a_0(x) \in \{1, \ldots, p\}$ with $a_0(x) \equiv x \mod p$.

Gross-Koblitz Formula

Proposition.

- Given $p > 11$, there exist $G_1(x), G_2(x) \in \mathbb{Z}_p$ such that for any $m \in \mathbb{Z}_p$,

$$\Gamma_p(x + mp) \equiv \Gamma_p(x) \left[1 + G_1(x)mp + G_2(x)\frac{(mp)^2}{2}\right] \pmod{p^3}.$$

- $G_1(x) = G_1(1 - x)$ and $G_2(x) + G_2(1 - x) = 2G_1(x)^2$.

Gross-Koblitz Formula. Let $\varphi : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ be the Teichmüller character such that $\varphi(x) \equiv x \pmod{p}$. Then

$$g\left(\varphi^{-j}\right) = -\pi_p^j \Gamma_p\left(\frac{j}{p-1}\right),$$

where $0 \le j \le p - 2$, and $\pi_p \in \mathbb{C}_p$ is a root of $x^{p-1} + p = 0$.

Example.

$$p \cdot {}_2F_1\begin{pmatrix} \eta_2, & \eta_2 \\ & \varepsilon \end{pmatrix}; -1 \Big)_p \equiv -\frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)}{\Gamma_p\left(\frac{3}{4}\right)} \pmod{p}.$$

Proof.

By the relations

$$p \cdot {}_2F_1\left(\begin{matrix} \eta_2, & \eta_2 \\ & \varepsilon \end{matrix}; -1\right)_p = J(\eta_4, \eta_2) + J(\overline{\eta_4}, \eta_2) = \frac{g(\eta_2)\left(g(\eta_4)^2 + g(\overline{\eta_4})^2\right)}{g(\overline{\eta_4})g(\eta_4)},$$

using the Gross-Koblitz formula, we see that

$$p \cdot {}_2F_1\left(\begin{matrix} \eta_2, & \eta_2 \\ & \varepsilon \end{matrix}; -1\right)_p = \frac{-\pi_p^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{2}\right)\left(\pi_p^{3\frac{p-1}{2}}\Gamma_p\left(\frac{3}{4}\right)^2 + \pi_p^{\frac{p-1}{2}}\Gamma_p\left(\frac{1}{4}\right)^2\right)}{\pi_p^{p-1}\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}$$

$$= -\frac{\Gamma_p\left(\frac{1}{2}\right)\left(-p\Gamma_p\left(\frac{3}{4}\right)^2 + \Gamma_p\left(\frac{1}{4}\right)^2\right)}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}.$$

Proposition. For any prime $p \equiv 1 \pmod 4$,

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix} ; -1\right]_{\frac{p-1}{2}} \equiv -\frac{\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{3}{4})} \pmod{p^2}. $$

Ideas. For any $x_1, x_2, y \in \mathbb{Z}_p$, we have

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2}+x_1p & \frac{1}{2}+x_2p \\ & 1+yp\end{matrix} ; -1\right]_{\frac{p-1}{2}} $$
$$ \equiv {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix} ; -1\right]_{\frac{p-1}{2}} + (x_1+x_2)Ap - yBp \pmod{p^2} $$

with $A = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k^2}{k!^2}\right) \cdot (-1)^k 2H_k^{(2)}$, and $B = \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{(\frac{1}{2})_k^2}{k!^2}\right) (-1)^k H_k$,

where $H_k^{(2)} := \sum_{j=1}^{k} \frac{1}{2j-1}$ and $H_k := \sum_{j=1}^{k} \frac{1}{j}$ are harmonic sums.

Ideas.

- $${}_2F_1\left[\begin{matrix} a & b \\ & a-b+1 \end{matrix};\ -1\right] = \frac{\Gamma(a-b+1)\Gamma(a/2+1)}{\Gamma(a+1)\Gamma(a/2-b+1)} = \frac{(a+1)_{-b}}{(a/2+1)_{-b}}$$
- When $b = \frac{1-p}{2}$,

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix};\ -1\right]_{\frac{p-1}{2}} + (x_1+x_2)Ap - (x_1-x_2)Bp \\ &\equiv \frac{\left(\frac{3}{2}+x_1p\right)_{-b}}{\left(\frac{5}{4}+\frac{x_1p}{2}\right)_{-b}} \pmod{p^2}, \end{aligned}$$

  a quotient of Γ_p-values.
- $\Gamma_p(\alpha+mp) \equiv \Gamma_p(\alpha)[1+G_1(\alpha)mp] \pmod{p^2}$, and $G_1(\alpha) = G_1(1-\alpha)$.

Example. If we let $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{2}$, ($b = \frac{1-p}{2}$), we have

$$\frac{\left(\frac{3}{2} + x_1 p\right)_{-b}}{\left(\frac{5}{4} + \frac{x_1 p}{2}\right)_{-b}} = \frac{\left(\frac{3+p}{2}\right)_{\frac{p-1}{2}}}{\left(\frac{5+p}{4}\right)_{\frac{p-1}{2}}} = -\frac{\Gamma_p(p)\Gamma_p(\frac{1}{4}+\frac{p}{4})}{\Gamma_p(\frac{1}{2}+\frac{p}{2})\Gamma_p(\frac{3}{4}+\frac{3p}{4})}.$$

Thus,

$$-\frac{\Gamma_p(p)\Gamma_p(\frac{1}{4}+\frac{p}{4})}{\Gamma_p(\frac{1}{2}+\frac{p}{2})\Gamma_p(\frac{3}{4}+\frac{3p}{4})}$$

$$\equiv -\frac{\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{3}{4})}\left[1 + G_1(0)p - G_1\left(\frac{1}{2}\right)\frac{p}{2} - G_1\left(\frac{1}{4}\right)\frac{p}{2}\right] \pmod{p^2},$$

$${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\ -1\right]_{\frac{p-1}{2}} - Bp$$

$$\equiv -\frac{\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{3}{4})}\left[1 + G_1(0)p - G_1\left(\frac{1}{2}\right)\frac{p}{2} - G_1\left(\frac{1}{4}\right)\frac{p}{2}\right] \pmod{p^2}.$$

Theorem.

$$ {}_{n+1}F_n\left[\begin{matrix} \frac{n-1}{n} & \frac{n-1}{n} & \cdots & \frac{n-1}{n} \\ & 1 & \cdots & 1 \end{matrix} ; 1\right]_{p-1} \equiv -\Gamma_p\left(\frac{1}{n}\right)^n \pmod{p^2}. $$

Idea. Use the special case of Karlsson–Minton formula:

$$ {}_{n+1}F_n\left[\begin{matrix} 1-p & 1+m+yp & 1+m & \cdots & 1+m \\ & 1+yp & 1 & \cdots & 1 \end{matrix} ; 1\right] = \frac{(-1)^{p-1}(p-1)!}{(1+yp)_m(m!)^{n-1}} = \frac{(p-1)!}{(1+yp)_m(m!)^{n-1}}. $$

Theorem. For each prime $p \equiv 1 \pmod 4$,

$$ {}_4F_3\left[\begin{matrix}\frac14 & \frac14 & \frac14 & \frac14 \\ & 1 & 1 & 1\end{matrix}; 1\right]_{p-1} \equiv (-1)^{\frac{p-1}{4}}\Gamma_p\left(\frac12\right)\Gamma_p\left(\frac14\right)^6 \pmod{p^4}. $$

Dougall.

$$ {}_7F_6\left[\begin{matrix}a & a/2+1 & b & c & d & e & -m \\ & a/2 & 1+a-b & 1+a-c & 1+a-d & 1+a-e & 1+a+m\end{matrix}; 1\right] $$
$$ = \frac{(1+a)_m(1+a-b-c)_m(1+a-b-d)_m(1+a-c-d)_m}{(1+a-b)_m(1+a-c)_m(1+a-d)_m(1+a-b-c-d)_m}. $$

Put

$$ a = 1/4, b = 5/8, c = 1/8, d = (1+pu)/4, $$
$$ e = (1+(1-u)p)/4, m = (p-1)/4). $$

Theorem. For each prime $p \equiv 1 \pmod 5$,

$$ {}_5F_4\left[\begin{matrix} \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ & 1 & 1 & 1 & 1 \end{matrix} ; 1\right]_{p-1} \equiv -\Gamma_p\left(\frac{1}{5}\right)^5 \Gamma_p\left(\frac{2}{5}\right)^5 \pmod{p^4}. $$

Conjecture.

$$ {}_5F_4\left[\begin{matrix} \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ & 1 & 1 & 1 & 1 \end{matrix} ; 1\right]_{p-1} \stackrel{?}{\equiv} -\Gamma_p\left(\frac{1}{5}\right)^5 \Gamma_p\left(\frac{2}{5}\right)^5 \pmod{p^5}. $$

Theorem. Let $q \equiv 1 \pmod 4$ be a prime power. Then

$$q^3 \cdot {}_4F_3\begin{pmatrix} \eta_4, & \eta_4, & \eta_4, & \eta_4 \\ & \varepsilon, & \varepsilon, & \varepsilon \end{pmatrix}; 1\Big)_q = J(\eta_4, \eta_2)^3 + qJ(\eta_4, \eta_2) - J(\overline{\eta_4}, \eta_2)^2.$$

McCarthy. For characters $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ in $\widehat{\mathbb{F}_q^\times}$, we define

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix}; x\right)^*_q := \frac{1}{q-1}\sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \prod_{i=0}^{n} \frac{g(A_i\chi)}{g(A_i)} \prod_{j=1}^{n} \frac{g(\overline{B_j\chi})}{g(\overline{B_j})} g(\overline{\chi})\chi(-1)^{n+1}\chi(x).$$

If $A_0 \neq \varepsilon$ and $A_i \neq B_i$ for each $1 \leq i \leq n$, then

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix}; x\right)_q = \left[\prod_{i=1}^{n} \binom{A_i}{B_i}\right] {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix}; x\right)^*_q .$$

McCarthy. For $A, B, C, D, E \in \widehat{\mathbb{F}_q^\times}$ such that, when A is a square, $A \neq \varepsilon$, $B \neq \varepsilon$, $B^2 \neq A$, $CD \neq A$, $CE \neq A$, $DE \neq A$, and $CDE \neq A$,

$$
{}_5F_4\begin{pmatrix} A, & B, & C, & D, & E \\ & A\overline{B}, & A\overline{C}, & A\overline{D} & A\overline{E}\end{pmatrix}; 1\Bigg)^*_q
= \frac{g(\overline{A})g(\overline{A}DE)g(\overline{A}CD)g(\overline{A}CE)}{g(\overline{A}C)g(\overline{A}D)g(\overline{A}E)g(\overline{A}CDE)} \sum_{R^2=A} {}_4F_3\begin{pmatrix} R\overline{B}, & C, & D, & E \\ & R & \overline{A}CDE, & A\overline{B}\end{pmatrix}; 1\Bigg)^*_q
+ \frac{g(\overline{A}DE)g(\overline{A}CD)g(\overline{A}CE)q}{g(C)g(D)g(E)g(\overline{A}C)g(\overline{A}D)g(\overline{A}E)} {}_2F_1\begin{pmatrix} A, & B \\ & A\overline{B}\end{pmatrix}; -1\Bigg)^*_q
$$

Whipple. If one of $1+\frac{1}{2}a-b$, c, d, e is a negative integer, then

$$
{}_5F_4\left[\begin{matrix} a & b & c & d & e \\ & 1+a-b & 1+a-c & 1+a-d & 1+a-e\end{matrix}; 1\right]
= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)}
\cdot {}_4F_3\left[\begin{matrix} 1+\frac{1}{2}a-b & c & d & e \\ & 1+\frac{1}{2}a & c+d+e-a & 1+a-b\end{matrix}; 1\right].
$$

Lemma.

- Let $q = p^e \equiv 1 \pmod 8$ be a prime power, and η_8 a character of order 8 in $\widehat{\mathbb{F}_q^\times}$ with $\eta_8^2 = \eta_4$. Then

$$q^4 \cdot {}_4F_3\begin{pmatrix} \eta_4, & \eta_4, & \eta_4, & \eta_4 \\ & \varepsilon, & \varepsilon, & \varepsilon \end{pmatrix}; 1\Big)_q = J(\eta_8,\eta_8)^4 - q \cdot {}_5F_4\begin{pmatrix} \eta_4, & \eta_4, & \eta_4, & \eta_4, & \eta_8 \\ & \varepsilon, & \varepsilon, & \varepsilon, & \eta_8 \end{pmatrix}; 1\Big)_q^*.$$

- Let $q = p^e \equiv 1 \pmod 8$ be a prime power. Then

$${}_5F_4\begin{pmatrix} \eta_4, & \eta_4, & \eta_4, & \eta_4, & \eta_8 \\ & \varepsilon, & \varepsilon, & \varepsilon, & \eta_8 \end{pmatrix}; 1\Big)_q^* = \frac{J(\eta_8,\eta_8)^4}{q} - qJ(\eta_4,\eta_2) - J(\eta_2,\eta_4)^3 + J(\eta_2,\overline{\eta}_4)^2.$$

Some Conjectures

- For any integer $n > 1$ and prime $p \equiv 1 \pmod{n}$,

$$ {}_3F_2 \left[\begin{matrix} \frac{1}{n} & \frac{1}{n} & \frac{n-1}{n} \\ & 1 & 1 \end{matrix} ; 1 \right]_{p-1} \equiv a_p(f_n(z)) \pmod{p^2}, $$

  where $a_p(f_n(z))$ is the pth coefficient of $f_n(z) = \sqrt[n]{E_1(z)^{n-1}E_2(z)}$ when expanded in terms of the local uniformizer $e^{2\pi i z/5n}$, and $E_1(z)$ and $E_2(z)$ are two explicit level 5 weight-3 noncongruence Eisenstein series with coefficients in $\mathbb{Z}$.
- For an integer $n > 2$, and any prime $p \equiv 1 \pmod{n}$,

$$ \sum_{k=0}^{p-1} \left(\frac{k!p}{\left(\frac{1}{n}+1\right)_k} \right)^n \equiv \sum_{k=\frac{(p-1)}{n}}^{p-1} \left(\frac{k!p}{\left(\frac{1}{n}+1\right)_k} \right)^n \stackrel{?}{\equiv} -\Gamma_p\left(\frac{1}{n}\right)^n \pmod{p^3}. $$