Hyperderivatives of Periods of Drinfeld Modules and Transcendence

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Notation

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\begin{aligned}
\mathbb{F}_q &:= \text{finite field, with } q \text{ a power of a prime } p \\
A &:= \mathbb{F}_q[t], \text{the polynomial ring in the variable } t \text{ over } \mathbb{F}_q \\
k &:= \mathbb{F}_q(\theta) = \text{rational functions in the variable } \theta \text{ over } \mathbb{F}_q \\
k_\infty &:= \mathbb{F}_q((1/\theta)), \text{the completion of } k \text{ with respect to } |\cdot|_\infty \\
\overline{k_\infty} &= \text{algebraic closure of } k_\infty \\
\bar{k} &:= \text{the algebraic closure of } k \text{ in } \overline{k_\infty} \\
\mathbb{K} &:= \text{completion of } \overline{k_\infty} \\
\mathbb{T} &:= \text{the Tate algebra of } \mathbb{K}[\![t]\!] \text{ on the closed unit disk} \\
\mathbb{L} &:= \text{fraction field of } \mathbb{T} \\
\mathrm{GL}_r/F &:= \text{for the field } F, \text{ the } F\text{-group scheme of invertible } r \times r \text{ matrices.}
\end{aligned}
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- For $f \in \mathbb{K}$ and $i \geq 1$, we define

$$\sigma^i(f) = f^{1/q^i} := f^{(-i)}$$

- We set $\tau = \sigma^{-1}$, that is

$$\tau^i(f) = f^{q^i} := f^{(i)}$$

for all $f \in \mathbb{K}$ and $i \geq 1$

Definition

Let t be a variable over $\mathbb{F}_q$ independent from θ. We define $\overline{k}(t)[\sigma, \sigma^{-1}]$ to be the polynomial ring in t and σ subject to the relations

$$at = ta, \quad \sigma t = t\sigma, \quad \sigma a = a^{1/q}\sigma, \qquad a \in \overline{k}$$

t-motives

Definition

A *pre-t-motive* M is a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module that is finite dimensional over $\overline{k}(t)$.

- Picking $\{m_1, \ldots, m_r\}$ a $\overline{k}(t)$-basis of M, we set $\boldsymbol{m} = [m_1, \ldots, m_r]^\top$.
- There is a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ such that

$$\sigma \boldsymbol{m} = \Phi \boldsymbol{m}$$

- M is *rigid analytically trivial* if there exists $\Psi \in \mathrm{GL}_r(\mathbb{L})$ so that

$$\Phi\Psi = \Psi^{(-1)}$$

- For $f \otimes m \in \mathbb{L} \otimes_{\overline{k}(t)} M$, we define

$$\sigma(f \otimes m) = f^{(-1)} \otimes \sigma m$$

We let M^B be the $\mathbb{F}_q(t)$-subspace fixed by σ.

- The category of rigid analytically trivial pre-t-motives, denoted by $\mathcal{R}$, is a neutral Tannakian Category over $\mathbb{F}_q(t)$ with fiber functor $M \mapsto M^B$.
- For $M \in \mathcal{R}$, we denote by $\mathcal{R}_M$, the strictly full Tannakian subcategory generated by M.
  Then,
  $$\mathcal{R}_M \approx \mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t))$$
  that is, $\mathcal{R}_M$ is equivalent to the category of representations over $\mathbb{F}_q(t)$ of an affine group scheme Γ_M over $\mathbb{F}_q(t)$.

Definition

An *Anderson t-motive* is a left $\overline{k}[t,\sigma]$-module $\mathcal{M}$ such that

- $\mathcal{M}$ is free and finitely generated as a $\overline{k}[t]$-module;
- $\mathcal{M}$ is free and finitely generated as a $\overline{k}[\sigma]$-module;
- $(t-\theta)^s\big(\mathcal{M}/\sigma\mathcal{M}\big) = \{0\}$ for some $s \in \mathbb{N}$.

- The functor
$$\mathcal{M} \mapsto \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}$$
from Anderson t-motives to pre-t-motives is fully faithful.
- We denote by $\mathcal{T}$, the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the essential image of rigid analytically trivial Anderson t-motives under the above functor. We call an element of $\mathcal{T}$, a *t-motive*.

Theorem (Papanikolas, 2008)

Let M be a t-motive and Γ_M be its Galois Group. If Φ represents multiplication by σ on M and Ψ its rigid analytic trivialization, then $tr.deg_{\overline{k}}\overline{k}(\Psi(\theta)) = dim\ \Gamma_M$

Anderson t-modules

Definition

A t-module over $\overline{k}$ is an $\mathbb{F}_q$-linear homomorphism $\phi : A \to \mathrm{Mat}_d(\overline{k}[\tau])$ with

$$\phi_t = B_0 + B_1\tau + \cdots + B_\ell \tau^\ell$$

where $B_i \in \mathrm{Mat}_d(\overline{k})$ with $\ell > 0$ and $B_0 = (\theta I_d + N)$ where I_d is the $d \times d$ identity matrix and N is a nilpotent matrix.

- For every t-module ϕ, there exists a unique exponential function

$$\mathrm{Exp}_\phi : \mathrm{Mat}_{d\times 1}(\mathbb{K}) \to \mathrm{Mat}_{d\times 1}(\mathbb{K})$$

  where $\mathrm{Exp}_\phi := \sum_{h=0}^{\infty} C_h \tau^h \in \mathrm{Mat}_d(\mathbb{K}[\![\tau]\!])$ satisfying

$$\mathrm{Exp}_\phi\big(B_0 \boldsymbol{z}\big) = \phi(t)\mathrm{Exp}_\phi\big(\boldsymbol{z}\big) \quad \text{and} \quad C_0 = I_d$$

- If Exp_ϕ is surjective, then we say that ϕ is *uniformizable*.
- A t-module with $d = 1$ is called a *Drinfeld Module*.

- We denote by Λ_ϕ, the kernel of Exp_ϕ, which is a discrete and finitely generated $\mathbb{F}_q[\theta]$-submodule of $\mathbb{K}^d$.

Definition

For $\boldsymbol{u} \in \mathbb{K}^d$, we define the *Anderson generating function for* ϕ to be

$$G_{\boldsymbol{u}}(t) := \sum_{n=0}^{\infty} \mathrm{Exp}_\phi(B_0^{-(n+1)}\boldsymbol{u})t^n \;\in \mathrm{Mat}_{1\times d}(\mathbb{K}[\![t]\!])$$

Let ρ be a Drinfeld module over $\overline{k}$ such that

$$\rho(t) = \theta + \kappa_1\tau + \cdots + \kappa_{r-1}\tau^{r-1} + \tau^r$$

r is the rank of ρ. We define a rigid analytically trivial Anderson t-motive associated to ρ in the following way:

- We let $\mathcal{M}_\rho$ be isomorphic to $(\overline{k}[t])^r$.
- if $\{m_1, \ldots, m_r\} \subset \mathcal{M}_\rho$ is the standard basis, we define multiplication by σ on $[m_1 \ldots m_r]^\top$ by

$$\begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ (t-\theta) & -\kappa_1^{(-1)} & \ldots & -\kappa_{r-1}^{-r+1} \end{pmatrix}$$

- Let $\{\lambda_1, \ldots, \lambda_r\}$ be an $\mathbb{F}_q[\theta]$-basis of Λ_ρ.
- If we let $f_u(t)$ denote the Anderson generating function for ρ, then the rigid analytic trivialization matrix is

$$\Psi_\rho = V^{-1}(\Upsilon^{(1)})^{-1}$$

where $V = \begin{pmatrix} \kappa_1 & \kappa_2^{(-1)} & \ldots & \kappa_{r-1}^{(-r+2)} & 1 \\ \kappa_2 & \kappa_3^{(-1)} & \ldots & 1 & \\ \vdots & \vdots & & & \\ \kappa_{r-1} & 1 & & & \\ 1 & & & & \end{pmatrix}$ and

$\Upsilon = \begin{pmatrix} f_1 & f_1^{(1)} & \ldots & f_1^{(r-1)} \\ f_2 & f_2^{(1)} & \ldots & f_2^{(r-1)} \\ \vdots & \vdots & & \\ f_r & f_r^{(1)} & \ldots & f_r^{(r-1)} \end{pmatrix}$ with $f_{\lambda_i}(t)$ denoted by $f_i(t)$.

- We let $M_\rho := \overline{k}(t) \otimes_{\overline{k}[t]} \mathcal{M}_\rho$ be the pre-t-motive associated to $\mathcal{M}_\rho$.

Theorem (Chang, Papanikolas, 2012)

Let ρ be a Drinfeld module of rank r defined over $\overline{k}$. Set $\mathcal{K}_\rho := End_{\mathcal{T}}(M_\rho)$ and define $Cent_{GL_r/\mathbb{F}_q(t)}(\mathcal{K}_\rho)$ to be the algebraic group over $\mathbb{F}_q(t)$ such that for any $\mathbb{F}_q(t)$-algebra R,

$$\mathrm{Cent}_{GL_r/\mathbb{F}_q(t)}(\mathcal{K}_\rho) := \{\gamma \in \mathrm{GL}_r(R) \mid \gamma g = g\gamma \text{ for all } g \in R \otimes_{\mathbb{F}_q(t)} \mathcal{K}_\rho\}$$

Then, $\Gamma_{M_\rho} = Cent_{GL_r/\mathbb{F}_q(t)}(\mathcal{K}_\rho)$.

Here $\mathcal{K}_\rho$ embeds naturally into $\mathrm{Mat}_r(\mathbb{F}_q(t))$. So, $R \otimes_{\mathbb{F}_q(t)} \mathcal{K}_\rho \subset \mathrm{Mat}_r(R)$.

Hyperderivatives and Hyperdifferential operators

For $m, j \geq 0$, we let

$$\binom{-m}{j} = (-1)^j \binom{m+j-1}{j}$$

Definition

Let F be a field with characteristic $p > 0$. For $j \geq 0$, we define the *j-th hyperdifferential operator* with respect to t

$$\partial_t^j : F(\!(t)\!) \to F(\!(t)\!)$$

to be the F-linear map on laurent series satisfying

$$\partial_t^j(t^m) = \binom{m}{j} t^{m-j}$$

where $m \in \mathbb{Z}$.

For $f \in F(\!(t)\!)$, we call $\partial_t^j(f)$ the *hyperderivative of f* with respect to t.

- (Due to Maurischat, 2017) Let ρ be a Drinfeld module. Then, we define a t-module $\mathrm{P}\rho$ given by

$$\mathrm{P}\rho(t) = \begin{pmatrix} \rho(t) & 0 \\ -1 & \rho(t) \end{pmatrix}$$

- For Φ_ρ and Ψ_ρ corresponding to $\mathcal{M}_\rho$, the Anderson t-motive associated with ρ, we have

$$\Phi_{\mathrm{P}\rho} = \begin{pmatrix} \Phi_\rho & \partial_t^1(\Phi_\rho) \\ 0 & \Phi_\rho \end{pmatrix}$$

and

$$\Psi_{\mathrm{P}\rho} = \begin{pmatrix} \Psi_\rho & \partial_t^1(\Psi_\rho) \\ 0 & \Psi_\rho \end{pmatrix}$$

where we take entrywise hyperderivative, corresponding to the Anderson t-motive $\mathcal{N}$ of $\mathrm{P}\rho$.

Theorem (N.)

Let ρ be a Drinfeld module of rank r defined over $\overline{k}$ and $\mathrm{P}\rho$ be its first prolongation t-module. If M_ρ and N are the t-motives corresponding to ρ and $\mathrm{P}\rho$ respectively, then $\dim \Gamma_N = 2 \cdot \dim \Gamma_{M_\rho}$.

- $\Gamma_N \cong \Gamma_{M_\rho} \ltimes W$ with $W \subset \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \; b \in \mathrm{Cent}_{\mathrm{Mat}_{r/\mathbb{F}_q(t)}}(\mathcal{K}_\rho) \right\}$
- We show $W = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathrm{Cent}_{\mathrm{Mat}_{r/\mathbb{F}_q(t)}}(\mathcal{K}_\rho) \right\}$

THANK YOU!