Multiple Polylogarithms over Tate Algebras

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Multiple zeta values in Classical Setting

Definition

For positive integers $s_1, \ldots, s_r$ with $s_1 \geq 2$, the multiple zeta value $\zeta(s_1, \ldots, s_r)$ is defined by Euler as the following infinite sum:

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} \in \mathbb{R}.$$

- In the definition, r is called the *depth* and $\sum_{i=1}^{r} s_i$ is called the *weight* of the multiple zeta value.
- It is clear from the definition that $\zeta(s_1, \ldots, s_r)$ is non-zero.
- (Euler's Reflection Formula) For any $s_1, s_2 > 1$, we have

$$\zeta(s_1, s_2) + \zeta(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2).$$

Some Notations for Function Field Setting

$$\begin{aligned}
q &:= \text{a positive power of a prime } p \\
\mathbb{F}_q &:= \text{finite field with } q \text{ elements} \\
A &:= \mathbb{F}_q[\theta] \\
A_+ &:= \text{the set of monic polynomials of } A \\
A_{+,d} &:= \text{the set of monic polynomials of } A \text{ of degree } d \\
K &:= \mathbb{F}_q(\theta) = \text{rational functions in the variable } \theta \text{ over } \mathbb{F}_q \\
\mathbb{K}_\infty &:= \mathbb{F}_q((1/\theta)) = \infty\text{-adic completion of } K \\
\mathbb{C}_\infty &:= \text{completion of the algebraic closure of } \mathbb{K}_\infty \\
|\cdot|_\infty &:= \text{the } \infty\text{-adic norm on } \mathbb{C}_\infty\text{, normalized so that } |\theta|_\infty = q
\end{aligned}$$

The Non-Commutative Rings $R\{\{\tau\}\}$ and $R\{\tau\}$

Definition

Let R be an $\mathbb{F}_q$-algebra and $\tau : R \to R$ be an $\mathbb{F}_q$-algebra homomorphism. We define the twisted power series ring $R\{\{\tau\}\}$ by the rule

$$\tau f = f^q \tau, \quad \forall f \in R,$$

and the twisted polynomial ring $R\{\tau\}$ is a subring.

- The ring $R\{\tau\}$ operates on R by setting for $\Delta = b_0 + \cdots + b_r\tau^r \in R\{\tau\}$ and $f \in R$,

$$\Delta(f) = b_0 f + b_1\tau(f) + \cdots + b_r\tau^r(f).$$

The Carlitz Module

Definition

The Carlitz module C (over A) is the $\mathbb{F}_q$-algebra homomorphism

$$\phi\colon A \to A\{\tau\}$$

defined by

$$C_\theta = \theta + \tau. \tag{1.1}$$

- As a function $C_\theta : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we define $C_\theta(x) = \theta x + x^q$ for all $x \in \mathbb{C}_\infty$.
- It also gives an A-module structure on $\mathbb{C}_\infty$ defined as

$$a \cdot x = C_a(x)$$

for all $a \in A$ and $x \in \mathbb{C}_\infty$.

The Polylogarithm Function

- Let $\ell_0 = 1$ and $\ell_i := (\theta - \theta^q) \dots (\theta - \theta^{q^i})$ for $i \geq 1$. The logarithm series corresponding to C is defined by

$$\log_C = \sum_{j \geq 0} \frac{1}{\ell_j} \tau^j \in K\{\{\tau\}\}$$

so that $\gamma_0 = 1$ and $\log_C C_a = a \log_C$ for all $a \in A$. It induces to the logarithm function $\log_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ which is defined by

$$\log_C(z) = \sum_{j \geq 0} \frac{1}{\ell_j} z^{q^j}$$

for all $x \in \mathbb{C}_\infty$ within the radius of convergence of $\log_\phi$.

- For any $n \in \mathbb{N}$, the *n-th polylogarithm function* $\log_n : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is defined by

$$\log_{n,C}(z) = \sum_{j \geq 0} \frac{1}{\ell_j^n} z^{q^j}.$$

Power Sums

- For integers $d, n \geq 0$, the power sum $S_d(n)$ is defined by

$$S_d(n) := \sum_{a \in A_{+,d}} \frac{1}{a^n}.$$

- For any integer $n \geq 1$, *the Carlitz-Goss zeta value* $\zeta_C(n)$ is given by

$$\zeta_C(n) = \sum_{d=0}^{\infty} S_d(n) = \sum_{a \in A_+} \frac{1}{a^n}.$$

- We define $D_0 := 1$ and $D_i = (\theta^{q^i} - \theta)D_{i-1}^q$ for $i \geq 1$. Moreover, for any positive integer n, assume that $n = \sum_{i=0}^{\infty} n_i q^i$ is the q-adic expansion of n. We define *the Carlitz factorial* Γ_n by

$$\Gamma_n := \prod_{i=0}^{\infty} D_i^{n_i}.$$

Anderson-Thakur Polynomials

Theorem (Anderson, Thakur, 1990)

For any integer $n \geq 0$, there exists a unique polynomial $H_n(t) \in A[t]$ such that for any $d \geq 0$ we have

$$S_d(n) = \sum_{a \in A_{+,d}} \frac{1}{a^n} = \frac{1}{\Gamma_n} . \frac{H_n(\theta^{q^d})}{\ell_d^n}.$$

Moreover, $\deg_t(H_n) < \frac{nq}{q-1}$.

Theorem (Anderson,Thakur, 1990)

For any integer $n \geq 0$, there exist elements $h_j \in A$ such that

$$\zeta_C(n) = \frac{1}{\Gamma_n} \sum_{j=0}^{m} h_j \log_{n,C}(\theta^j).$$

Multiple zeta values on Function Fields

- (Thakur) For any r-tuple $(s_1, \ldots, s_r) \in \mathbb{Z}_{\geq 0}$, we define the multiple zeta value $\zeta_C(s_1, \ldots, s_r)$ by

$$\zeta_C(s_1, \ldots, s_r) = \sum_{d_1 > d_2 > \cdots > d_r \geq 0} S_d(s_1) \ldots S_d(s_r) \quad (2.1)$$

$$= \sum_{\substack{\deg(a_1) > \deg(a_2) > \cdots > \deg(a_r) \geq 0 \\ a_i \in A_+}} \frac{1}{a_1^{s_1} \ldots a_r^{s_r}} \in \mathbb{K}_\infty. \quad (2.2)$$

- We call $\zeta_C(s_1, \ldots, s_r)$ the multiple zeta value of depth r and weight $\sum_{i=1}^r s_i$.

Some Properties of Multiple zeta values

Theorem (Thakur, 2009)

The multiple zeta value $\zeta_C(s_1, \dots, s_r)$, with $s_i > 0$ is always non-zero.

The sum-shuffle identity as in Euler's reflection formula fails in general. But Thakur proved the following identity.

Theorem (Thakur, 2009)

When $s_1, s_2 \leq q$ and $s_1 + s_2 > q$ we have

$$\zeta_C(s_1)\zeta_C(s_2) = \zeta_C(s_1 + s_2) + \zeta_C(s_1, s_2) + \zeta_C(s_2, s_1) + (s_1 + s_2 - q)\zeta_C(a + b - q + 1, q - 1).$$

Theorem (Thakur, 2010)

The product of multiple zeta values can be written as an $\mathbb{F}_p$- linear combination of some multiple zeta values. In particular, the $\mathbb{F}_p$-vector space generated by multiple zeta values is an $\mathbb{F}_p$-algebra.

Carlitz Multiple Polylogarithms

Definition

For any given r-tuple $(s_1, \ldots, s_r) \in \mathbb{Z}_{\geq 0}$, we define its associated *Carlitz multiple polylogarithm* as

$$\mathrm{Li}_{s_1,\ldots,s_r}(z_1, \ldots, z_r) = \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \ldots z_r^{q^{i_r}}}{\ell_{i_1}^{s_1} \ldots \ell_{i_r}^{s_r}}.$$

Theorem (Chang, 2014)

Let S be the set of points $u = (u_1, \ldots, u_r) \in A^r$ where u_j is any coefficient of the polynomial $H_{s_j}(t) \in A[t]$ for any $j \in \{1, \ldots, r\}$. For any $u \in S$, let u_j be the coefficient of t^{m_j} in $H_{s_j}(t)$ and let $a_u := \prod_{i=1}^{r} \theta^{m_i}$. Then we have

$$\zeta_C(s_1, \ldots, s_r) = \frac{1}{\Gamma_{s_1} \ldots \Gamma_{s_r}} \sum_{u \in S} a_u \, \mathrm{Li}_{s_1,s_2,\ldots,s_r}(u).$$

Introduction to Tate Algebras

- $\mathbb{T}_s = \{\sum a_{\nu_1\cdots\nu_s} t_1^{\nu_1}\cdots t_s^{\nu_s} \in \mathbb{C}_\infty[[t_1,\dots,t_n]] \mid \quad |a_{\nu_1\cdots\nu_s}|_\infty \to 0 \quad as \quad \nu_1+\cdots+\nu_s \to \infty\}$
- We define the Gauss norm $\|\cdot\|$ on $\mathbb{T}_s$ by setting for $f = \sum a_{\nu_1\cdots\nu_s} t_1^{\nu_1}\cdots t_s^{\nu_s} \in \mathbb{T}_s$,

$$\|f\| := \sup\{|a_{\nu_1\cdots\nu_s}|_\infty : \nu \in \mathbb{Z}_{\geqslant 0}^s\}.$$

  We note that $\mathbb{T}_s$ is complete with respect to $\|\cdot\|$.
- We define an automorphism $\tau : \mathbb{T}_s \to \mathbb{T}_s$ by

$$\tau\left(\sum_{\nu_i\geq 0} a_{\nu_1\cdots\nu_s} t_1^{\nu_1}\cdots t_s^{\nu_s}\right) := \sum_{\nu_i\geq 0} a_{\nu_1\cdots\nu_s}^q t_1^{\nu_1}\cdots t_s^{\nu_s}.$$

Twisted Power Sums over Tate Algebras

- For integers $d, n \geq 0$, the power sum $S_d(t_1, \ldots, t_s; n)$ is defined by

$$S_d(t_1, \ldots, t_s; n) := \sum_{a \in A_{+,d}} \frac{a(t_1) \ldots a(t_s)}{a^n}.$$

- Set $b_0(t_j) := 1$. For any integer $i \geq 1$, we introduce elements $b_i(t_j)$ by

$$b_i(t_j) := \prod_{k=0}^{i-1} (t_j - \theta^{q^k}).$$

Theorem (Demeslay, 2015)

For any integer $n \geq 0$ and $r \in \mathbb{Z}_{\geq 1}$ such that $q^r \geq n$, there exists a unique polynomial $Q_n(t) \in \mathbb{T}_n[t]$ such that for any $d \geq 0$ we have

$$S_d(t_1, \ldots, t_s; n) = \sum_{a \in A_{+,d}} \frac{a(t_1) \ldots a(t_s)}{a^n} = \frac{b_d(t_1) \ldots b_d(t_s)}{\ell_{r-1}^{q^r - n} b_r(t_1) \ldots b_r(t_s) \ell_d^n} . \tau^d (Q_n(t))_{|t=\theta}.$$

Moreover, $\|Q_n(t)\| < q^{\frac{nq-s}{q-1}}$.

Pellarin L-series and Polylogarithm Functions over Tate Algebras

- For any integer $N \geq 1$, *Pellarin L-series* $L(t_1, \dots, t_s; N)$ is defined by

$$L(t_1, \dots, t_s; N) := \sum_{a \in A_+} \frac{a(t_1) \dots a(t_s)}{a^N} \in \mathbb{T}_s.$$

- Let z be an independent variable over $\mathbb{C}_\infty$ and f be an element in $\mathbb{T}_s$. In a similar way to the definition of $\log_{n,C}$, we define

$$\log_{N,z}(f) = \sum_{i \geq 0} \frac{b_i(t_1) \dots b_i(t_s) z^i}{\ell_i^N} \tau^i(f) \in \mathbb{T}_{s+1}.$$

  where $\mathbb{T}_{s+1}$ is the Tate algebra with variables $t_1, \dots, t_s, z$.

- Furthermore, we set

$$\log_N(f) := \log_{N,z}(f)_{|z=1}.$$

Analogue of Anderson and Thakur's Result

Theorem (Anglès, Pellarin, Tavares Ribeiro, 2018)

For all integers $N \in \mathbb{Z}$, $s \geq 1$ and $r \geq 1$ such that $q^r \geq N$, there exists an integer $m \geq 0$, and for $0 \leq j \leq m$, polynomials $h_j(z) \in A[t_1, \ldots, t_s][z]$ such that

$$\sum_{d=0}^{\infty} z^d \sum_{a \in A_{+,d}} \frac{a(t_1) \ldots a(t_s)}{a^N} = \frac{1}{\ell_{r-1}^{q^r - N} b_r(t_1) \ldots b_r(t_s)} \sum_{j=0}^{m} \theta^j \log_{N,z}(h_j). \tag{4.1}$$

Corollary

The equality (4.1) *implies*

$$L(t_1, \ldots, t_s; N) = \sum_{a \in A_+} \frac{a(t_1) \ldots a(t_s)}{a^N}$$

$$= \frac{1}{\ell_{r-1}^{q^r - N} b_r(t_1) \ldots b_r(t_s)} \sum_{j=0}^{m} \theta^j \log_N(h_j(1)).$$

Multiple zeta values over Tate Algebras

- Let Σ be a subset of $\mathbb{Z}_{\geq 1}$ and for some $r \in \mathbb{N}$ and for $1 \leq i \leq r$, $U_i \subset \Sigma$ be the subsets such that $\Sigma = \sqcup_{i=1}^{r} U_i$. Moreover, let $\mathbb{T}_\Sigma$ be the Tate algebra with variables t_j for $j \in \Sigma$.
- Let $\mathbb{F}_q[\Sigma]$ be the polynomial ring in t_j for $j \in \Sigma$ with coefficients in $\mathbb{F}_q$. Define the function $\sigma_{U_i} : A_+ \to \mathbb{F}_q[\Sigma]$ by

$$\sigma_{U_i}(a) := \prod_{i \in U_i} a(t_i)$$

  for any $a \in A_+$.
- For any integer $d, n \geq 0$, we set $S_d(U_i, n)$ as the the following twisted power sum:

$$S_d(n, U_i) := \sum_{a \in A_{+,d}} \frac{\sigma_{U_i}(a)}{a^n} = \frac{\prod_{i \in U_i} b_d(t_i)}{\ell_d^n} . \tau^d (Q_{U_i,n}(t))_{|t=\theta},$$

  where $Q_{U_i,n}(t) \in \mathbb{T}_\Sigma[t]$ and the last equality follows from Demeslay's result.

- For any $s_i \in \mathbb{N}$ and for all $1 \leq i \leq r$ let us set the composition array $\mathcal{C}$ as

$$\mathcal{C} := \begin{pmatrix} \sigma_{U_1}, \sigma_{U_2}, \dots, \sigma_{U_r} \\ s_1, s_2, \dots, s_r \end{pmatrix}. \tag{5.1}$$

Definition (Pellarin)

For any $\mathcal{C}$ as in (5.1), we define multiple zeta values over Tate algebras as the following object:

$$\zeta_C(\mathcal{C}) := \sum_{\substack{\deg(a_1)>\deg(a_2)>\cdots>\deg(a_r)\geq 0 \\ a_i \in A_+}} \frac{\sigma_{U_1}(a_1)\sigma_{U_2}(a_2)\dots\sigma_{U_r}(a_r)}{a_1^{s_1} a_2^{s_2} \dots a_r^{s_r}}$$

$$= \sum_{i_1>i_2>\cdots>i_r\geq 0} S_{i_1}(s_1, U_1)\dots S_{i_r}(s_r, U_r) \in \mathbb{T}_\Sigma.$$

Some Properties of Multiple zeta values over Tate algebras

Theorem (Pellarin, 2017)

The multiple zeta value $\zeta_C(\mathcal{C})$ where $\mathcal{C}$ as in (5.1) is always non-zero.

We also have some sum-shuffle identities in this setting.

Theorem (Pellarin, 2017)

The following formula holds, for all $\Sigma \subset \mathbb{Z}_{\geq 1}$ and $U \sqcup V = \Sigma$:

$$\zeta_C\left(\begin{pmatrix} \sigma_U \\ 1 \end{pmatrix}\right)\zeta_C\left(\begin{pmatrix} \sigma_V \\ 1 \end{pmatrix}\right) - \zeta_C\left(\begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix}\right) = \zeta_C\left(\begin{pmatrix} \sigma_U, \sigma_V \\ 1, 1 \end{pmatrix}\right) + \zeta_C\left(\begin{pmatrix} \sigma_V, \sigma_U \\ 1, 1 \end{pmatrix}\right) - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} \zeta_C\left(\begin{pmatrix} \sigma_I, \sigma_J \\ 1, 1 \end{pmatrix}\right).$$

where $|J|$ denotes the number of elements in J.

- Furthermore, Pellarin showed that for any set $\Sigma \subset \mathbb{Z}_{\geq 1}$, the $\mathbb{F}_p$-vector space generated by multiple zeta values is an $\mathbb{F}_p$-algebra.

Multiple Polylogarithms over Tate Algebras

Remark

At this point, the natural direction would be understanding the multiple polylogarithms in this new setting.

Definition

For any $i, d \in \mathbb{N}$, set $b_d(U_i) := \prod_{j \in U_i} b_d(t_j)$. For any r-tuple $(f_1, \ldots, f_r) \in \mathbb{T}^r_{\Sigma,t}$, we set

$$\mathrm{Li}_{s_1,\ldots,s_r}(f_1,\ldots,f_r) := \sum_{i_1 > i_2 > \cdots > i_r \geq 0} \frac{\tau^{i_1}(f_1)\ldots\tau^{i_r}(f_r) b_{i_1}(U_1)\ldots b_{i_r}(U_r)}{\ell_{i_1}^{s_1}\ldots\ell_{i_r}^{s_r}}$$

as the multiple polylogarithms for multiple zeta values over Tate algebras.

Theorem

Let S be the set of points $u = (u_1, \dots, u_r) \in \mathbb{T}_\Sigma^r$ where u_j is any coefficient of the polynomial $Q_{U_j,s_j}(t) \in \mathbb{T}_\Sigma[t]$ for any $j \in \{1, \dots, r\}$. For any $u \in S$, let u_j be the coefficient of t^{m_j} in $Q_{U_j,s_j}(t)$ and let

$$a_u := \prod_{i=1}^{r} \theta^{m_i}.$$

We define $r_{s_i} \geq 1$ as the smallest integer such that $s_i \leq q^{r_{s_i}}$. Then we have

$$\zeta_C\begin{pmatrix} \sigma U_1, \sigma U_2, \dots, \sigma U_r \\ s_1, s_2, \dots, s_r \end{pmatrix} = \frac{1}{\prod_{i=1}^{r} \ell_{r_{s_i}-1}^{q^{r_{s_i}}-s_i} b_{r_{s_i}}(U_i)} \sum_{u \in S} a_u \operatorname{Li}_{s_1,s_2,\dots,s_r}(u).$$

Some Examples

Example

- Define $\Sigma = \{1, 2, \ldots, r\}$ and let $U_i = \{i\}$ for any $1 \leq i \leq r$. Then

$$\zeta_C\left(\begin{pmatrix}\sigma_{U_1}, \sigma_{U_2}, \ldots, \sigma_{U_r}\\ 1, 1, \ldots, 1\end{pmatrix}\right) = \sum_{\substack{\deg(a_1) > \deg(a_2) > \cdots > \deg(a_r) \geq 0\\ a_i \in A_+}} \frac{a_1(t_1) \ldots a_r(t_r)}{a_1^{s_1} a_2^{s_2} \ldots a_r^{s_r}}$$
$$= \mathrm{Li}(1, 1, \ldots, 1).$$

- Let Σ and U_i be as in the previous example and $1 < s_i \leq q$ for any $1 \leq i \leq r$. Then we have

$$\zeta_C\left(\begin{pmatrix}\sigma_{U_1}, \sigma_{U_2}, \ldots, \sigma_{U_r}\\ s_1, s_2, \ldots, s_r\end{pmatrix}\right) = \sum_{\substack{\deg(a_1) > \deg(a_2) > \cdots > \deg(a_r) \geq 0\\ a_i \in A_+}} \frac{a_1(t_1) \ldots a_r(t_r)}{a_1^{s_1} a_2^{s_2} \ldots a_r^{s_r}}$$
$$= \mathrm{Li}(t_1 - \theta, t_2 - \theta, \ldots, t_s - \theta).$$

THANK YOU !